Stability of Quasicrystal Frameworks in 2D and 3D

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Summary

We study the rigidity of quasicrystal rod and pinion frameworks. In collaboration with mathematical artist Tony Robbin we found the minimum number and positions of braced rhombi which stabilize 2D quasicrystal frameworks, which we call *Penrose Carpets*.

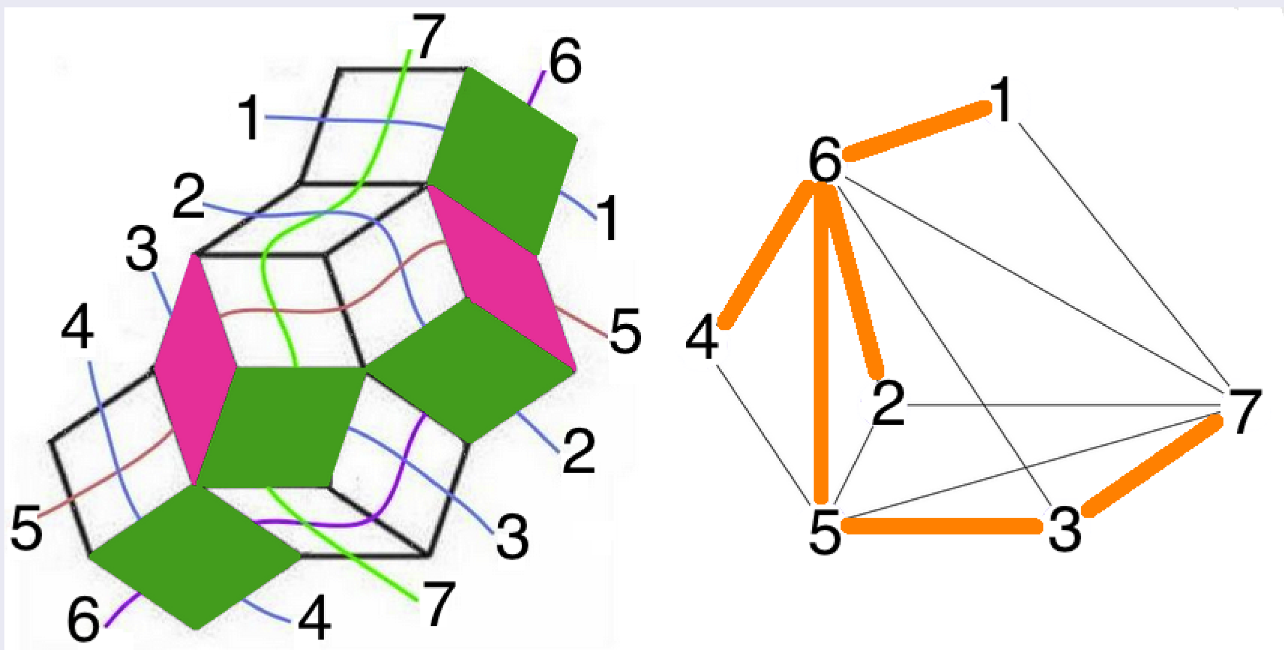
The Danish engineer Ture Wester conjectured the solution and we proved him right. We continue to work on the 3D case as originally proposed by Tony Robbin to help him build stable 3D quasicrystal frameworks, such as COAST.

Tony Robbin COAST.

Installed 1994, Danish Technical University. Destroyed!!

Definitions

The ribbons are numbered $1, 2, 3, 4, 5, 6, 7$. The rhombus where ribbons 1 and 7 cross is the black edge (17) in the Wester graph. The orange edges form a Wester tree, corresponding to the colored plates.



Wester's Theorem

Let F be a quasicrystal framework with associated Wester graph Γ and let B be a maximal subtree of Γ. Then the bracing corresponding to B makes F rigid. Moreover, the minimum number of rhombi needed to make F rigid is equal to the number of ribbons minus one.

Proof by the Wester Game

Playing the Wester Game with quasiIGLgraph.exe, our Real-time Interactive Computer Animation (RTICA) programmed in C++/OpenGL, motivates the proof of Wester's Theorem.

YOU Start by loading a 2D Penrose quasicrystal framework (CARPET).

HAL The computer calculates maximal sequences of contiguous rhombi which share one direction along parallel sides (RIBBONS).

YOU Navigate the carpet to choose and brace rhombi (PLATES).

HAL The database marks the plates as non-deformable.

YOU Choose successive ribbons and tell the computer to deform.

HAL The computer replies with a SHIFT associated with the ribbon thus:

HAL If the ribbon has no bracing plates then the parallel edges in the ribbon all turn a small angle (SHEAR), keeping one complementary component of the carpet fixed, and moving the any other component rigidly.

HAL If the ribbon has has one or more plates, then recursively, a shear along the ribbon requires a compensating shear in the other ribbon through all of its plates, to keep them rigid (SHIFT).

HAL The shift along a particular ribbon may effect a rigid rotation of the entire carpet without deforming any rhombi. If it does so along all ribbons, the carpet is undeformable except for uniform rotations (RIGID).

YOU But you have not won the game yet. Now remove plates, still keeping the carpet rigid. When you cannot remove anymore plates, what remains is a solution to the problem.

Generalizations

The above argument was first applied by Baglivo and Graver to a quadrille carpet, made of square rhombi. It generalizes to a variety of other shapes provided they have symmetry properties analogous to those of squares. Penrose rhombi have angles $i\frac{2\pi}{10}$ and $j = 5 - i$, and thus come in only two shapes, $i = 1, j = 4$ (FAT) and $i = 2, j = 3$ (SKINNY). Since braced rhombi can still rotate under shifts along ribbons, it is important to note that the orientation of a Penrose rhombus is determined by the orientation of one side, and its shape by one angle. Thus all plates in a braced connected subgraph are oriented coherently. If it is spanning, then both ribbons of an unbraced rhombus meet a plate. So the rhombus is also Penrose. A maximal tree is both connected, spanning and minimal.

As we all know, architectural structures made up of triangular framings, such as Buckminster Fuller domes, are stabilized by their tetrahedra. Quasicrystal frameworks are made of rombohedra, which have to be braced with rhombic plates, for instance. The 3D generalization of our proof works in part, and meets fromidable obstructions in others, because of the failure of some of these properties. We're working on it!

Bibliography

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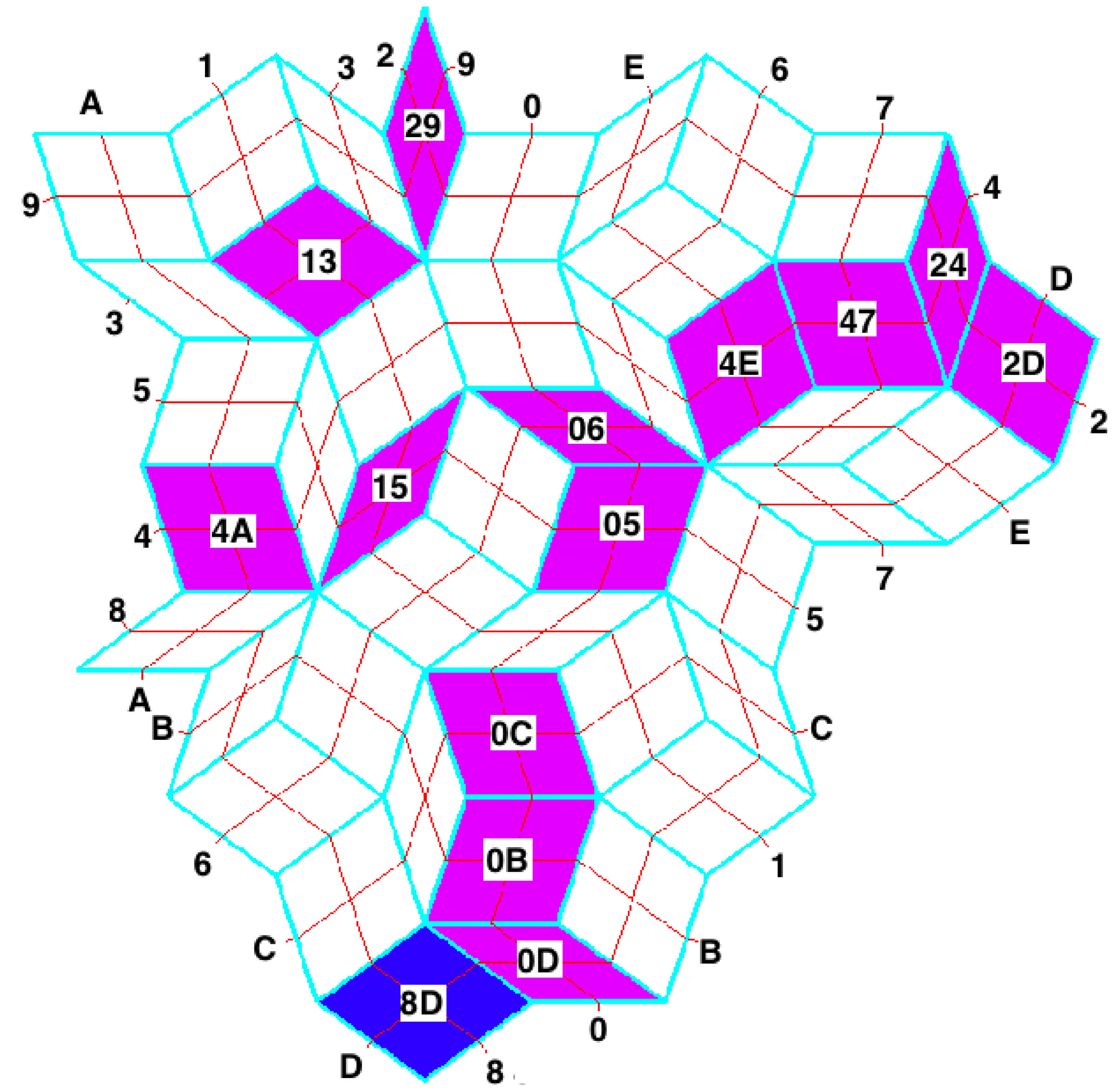
- Richeson, D 2010 *An application of graph theory to architecture*, http://divisbyzero.com/2010/03/14/an-application-of-graph-theory-to-architecture/
- Wester, Ture, 2006 *The Structural Morphology of Penrose and Quasicrystal Patterns Part I*, International Conference on Adaptable Building Structures.

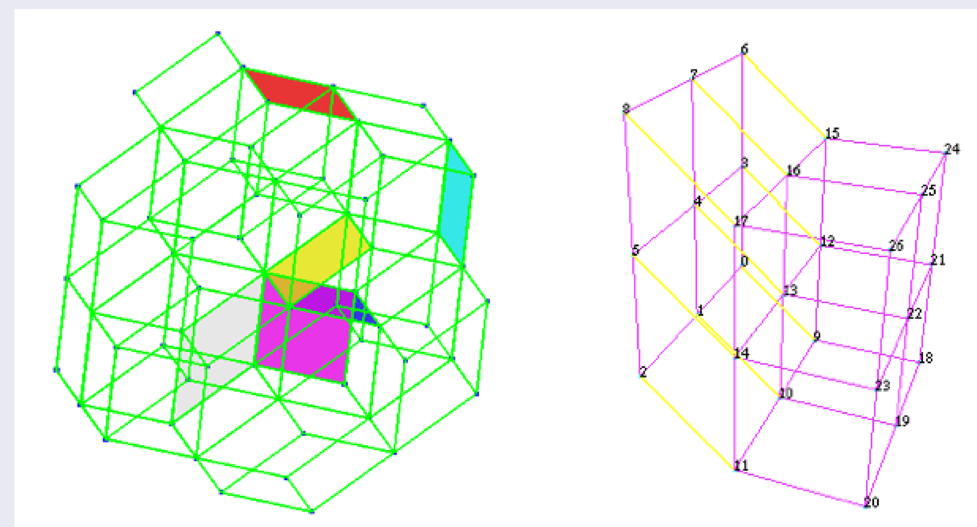
Figure for Wester's Theorem

Wester Tree Bracing of a Penrose Carpet with 15 Ribbons



The 3D Quasicrystal Frameworks

Han Chong has programmed two more RTCA, similar to his quasiIGLgraph.exe, to the 3D case.



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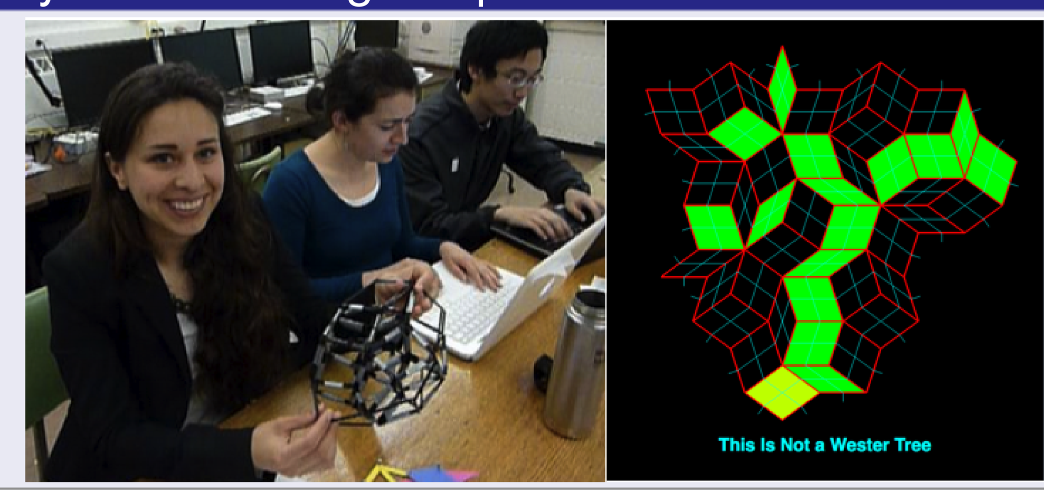

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