

## Stability of Quasicrystal Frameworks in 2D and 3D

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**Summary:** The solution to the problem of how to make a planar Penrose rod and pinion framework rigid in a minimal way was conjectured by Ture Wester [1]. In his paper, Wester states that the proof of his solution can be achieved by following the exposition of Baglivo and Graver [2] who deal with the case where the framework is a grid of squares. The interesting project of studying the stability of 3D quasicrystal frameworks was proposed to us by Tony Robbin [3] in a lecture on 26 November 2012. Our goal in this paper is to explain the essence of the Baglivo-Graver and Wester approach and prove a modest generalization of both. Their solution applies to any planar rod and pinion framework composed of quadrilaterals. We also present a real-time interactive computer animation designed to explore distortions of 2D Penrose frameworks. The problem of how to make an equilateral rod and pinion 3D quasicrystal framework rigid by introducing plates bracing certain faces of the rhombohedra is still under discussion. We continue to study 3D quasicrystal frameworks and their rigidity from Wester's point of view. However, their possible distortions can be so severe as to make them hard to analyze. For one, in 3D a twisting deformation of a rhombohedron is no longer a rhombohedron. Some of its faces become non-planar (skew) quadrilaterals. In our exposition of the 2D case we will indicate which properties do not generalize to 3D.

**Keywords:** *Quasicrystals, Rod and Pinion Framework, Rigidity, Penrose Tiling, Sub Graph, Spanning Tree.*

### RHOMBIC CARPETS

We study the rigidity of quasicrystal rod and pinion frameworks in the plane. That is, the rods (edges) cannot change length. Adjacent rods  $A, B$  at a pinion may rotate as long as their angle remains proper,  $0 < \angle AB < \pi$ . In collaboration with mathematical artist Tony Robbin, and University of Illinois students in the Illinois Geometry Lab (IGL), Alex Burnley and Chong Han, we tested Wester's proposed solution for the minimum number and positions of braced rhombi which stabilize a 2D framework which were derived from a finite portion of a Penrose aperiodic tiling of the plane. We shall refer to such a framework as a *Penrose carpet*.



Fig. 01 *Quasicrystal framework COAST* by Tony Robbin, installed at the Danish Technical University, 1994.

We found that Wester's solutions is mathematically justified provided additional assumptions are made. At the same time, we found the additional assumptions that the carpet is derived from a Penrose tiling, or is composed of squares as in [2], are not relevant to the argument.

### DEFINITION AND OBSERVATIONS

The fundamental fact that, in the plane, any 4-sided polygon with equal length sides must be a rhombus, leads us to consider *rhombic frameworks* in general. Endowed with the additional topological properties, we call them *rhombic carpets*. In particular, when the rhombi are squares, we shall call them *rectangular carpets* with Baglivo-Graver. When they have the shape of the thin and thick rhombi (with interior angles  $36^\circ, 144^\circ$  and  $72^\circ, 108^\circ$ ) considered by Wester, we shall refer to them as *Penrose carpets*.

A *rhombic carpet* is a planar arrangement of rhombi, which is *connected* and *simply connected*. This shall mean that every rhombus may be reached from any other by a contiguous succession of rhombi. Two rhombi are *contiguous* if they share a common side. Two diamonds touching at a vertex will not be considered as connected. By *simply connected* we shall mean that the carpet has no holes. That is, if two different rhombic paths connect the same two rhombi then the space between them is entirely filled by rhombi. This last condition is not necessary for every argument we make, but it makes the argument easier. Moreover, the minimal solutions obtained in their separate cases by Baglivo-Graver and Wester do require

this, as a counterexample will show. Finally, we shall not allow two rhombi to overlap.

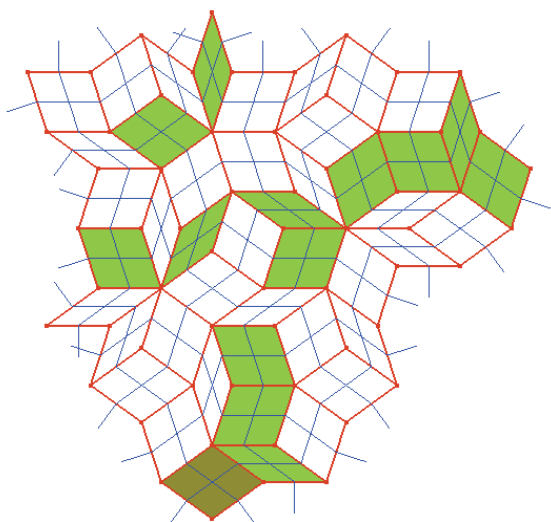


Fig. 02 Wester tree bracing of a Penrose carpet with 15 ribbons (blue) and 14 braced rhombi (green plates). Each ribbon has at least one plate. Two ribbons are connected by a unique succession of plates and ribbons.

By a *direction* in the plane we mean the family of all lines parallel to a given line, technically called a *pencil* of lines. For  $r, t$  two directions in the plane we can observe a pair of supplementary angles between them. Taking a line from each pencil, we denote the smaller of the two angles of intersection by  $\angle rt$ .

This angle is zero for parallel lines. In a rhombic carpet, a maximal succession of contiguous rhombi, whose common edges are all parallel, is called a *ribbon*. This pencil of parallels is called the *direction of the ribbon*. Note the technical use of the term. The “direction” taken by the ribbon is always “across” its direction in our sense.

Penrose and rectangular carpets both have the property that any two ribbons  $a, b$ , either have no rhombus in common (they are said to be *parallel ribbons*), or they intersect at exactly one rhombus, which we denote by  $(ab)$ . Since we do not allow rhombi to overlap and because they lie in the Euclidean plane, two crossing ribbons cannot recross on a second rhombus. Thus the notation  $(ab)$  makes sense.

It is important to note that the previous definitions continue to make sense for any *small distortion* of a rhombic carpet in the plane, namely one that does not destroy the topological properties we have required.

The only distortion a rhombus can undergo in the plane is a change in its angles. Although the two directions of its opposite sides may rotate, it remains a rhombus. This property does not generalize to the rhombohedra of 3D quasicrystal rod and pinion frameworks. Twisting distortions of rhombohedra cease to have parallel edges and faces.

## BRACING A RHOMBIC CARPET

We *brace* a rhombus  $(ab)$  in the carpet, for instance with a rigid plate filling the rhombus, to prevent it from changing its shape under a distortion of the carpet. Note that the angle  $\angle ab$  is *fixed* in that it is invariant under every small distortion of the carpet. Braced rhombi can only rotate and translate under a distortion of the carpet. The lines in the directions of  $a$  and  $b$  can rotate, but the angle between them is fixed by  $(ab)$ .

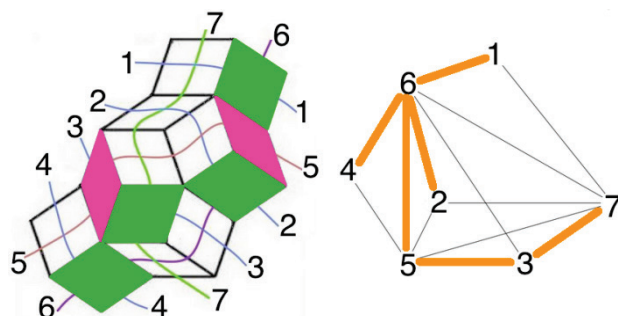


Fig. 03 Left: A Penrose carpet with 13 rhombi, 7 ribbons and 6 bracing plates. Right: the corresponding Wester graph and a maximal bracing tree.

We define a *rigid bracing* of a rhombic carpet to be one in which the interior angles of all of the rhombi in the carpet are fixed. For a rigidly braced Penrose carpet, the angles are multiples of  $36^\circ$ . For a rectangular carpet, they are right angles. In terms of the directions of the ribbons, in a rigidly bracing rhombic carpet, the angle  $\angle ab$  is fixed for every rhombus  $(ab)$ .

## WESTER'S THEOREM

So, when is a braced rhombic carpet rigid? Following Baglivo-Graver, and Wester, we associate to a rhombic carpet  $K$  a graph  $\Gamma$ , which we called the *Wester Graph* of the carpet. The vertices of  $\Gamma$  are all of the ribbons in the carpet. The edges are defined by the rule that  $ab$  is an edge if and only if the ribbons  $a$  and  $b$  intersect. Thus the edges of  $\Gamma$  correspond to the rhombi of carpet  $K$ . Next, a (partial) *bracing* of the carpet corresponds to a subgraph,  $\Phi$  of  $\Gamma$  consisting in edges  $ab$  whose rhombus  $(ab)$  in the carpet is braced. We shall refer to  $\Phi$  as a *bracing subgraph* of the Wester graph  $\Gamma$ .

We emphasize the transitivity property in the plane by announcing a critical fact:

### Transitivity Lemma

Let  $a, b, c$  be ribbons in a (partially braced) rhombic carpet. If  $\angle ab$  and  $\angle bc$  are fixed then  $\angle ac$  is also fixed.

**Proof:** Choose representative lines from each of the three directions in general position (i.e. not concurrent). Recall that we choose the acute angle where two lines cross,

$$\alpha = \angle bc, \quad \beta = \angle ac, \quad \gamma = \angle ab.$$

Because we are in the Euclidean plane, where the interior angles of a triangle add to  $\pi$ , in every case the three angles satisfy a linear relation to  $0$  or  $\pi$ . Thus, if two of these

angles remain invariant under a small distortion of the carpet in the plane, so does the third.  $\square$

Note, when thinking about the direction of a ribbon, it is useful to imagine the family of lines as being the extended sides between adjacent rhombi in the ribbon. Under a distortion, the direction of these lines may rotate, but they remain mutually parallel.

**Theorem**

Let  $K$  be a rhombic carpet with associated Wester graph  $\Gamma$  and let  $\Phi$  be a subgraph which is both spanning and connected. Then, bracing the rhombi corresponding to the edges of  $\Phi$  makes  $K$  rigid.

**Proof:** We want to show that if we have a subgraph  $\Phi$  which is connected and spanning, then the corresponding bracing of the carpet is rigid. To do this it is enough to show that under any attempt at distorting the carpet the angle  $\angle ab$  between two ribbons,  $a, b$  crossing on a rhombus  $(ab)$ , is fixed. In other words, all the distortion can achieve is a uniform rotation of the entire carpet. Let  $(ab)$  be any rhombus of the carpet. If it is one of the braced rhombi, then it will preserve its two angles under every distortion. So assume that the edge  $ab$  in the Wester graph is not in  $\Phi$ . Since  $\Phi$  is connected and spanning, there exists an edge-path in  $\Phi$  between the vertex  $a$  and  $b$ . We denote this path by its consecutive vertices :

$$a, r_1, r_2, r_3, \dots, r_n, b.$$

Since the path is in  $\Phi$ , all of the rhombi

$$(ar_1), (r_1r_2), (r_2r_3), \dots, (r_nb)$$

are braced. Which implies all of the angles

$$\angle ar_1, \angle r_1r_2, \angle r_2r_3, \dots, \angle r_nb$$

are fixed. Using the transitivity lemma for the ribbons  $a, r_1, r_2$  we conclude that  $\angle ar_2$  is fixed. Recursively, we see that  $\angle ab$  is fixed.  $\square$

It is easy to see that a maximal tree in a connected graph is spanning, connected and has one fewer edges than the number of vertices in the graph. We shall call such an acyclic subgraph a *Wester tree*. A cyclic edge path is called a *cycle*, and a tree has no cycles.

**Corollary:** If a bracing subgraph,  $\Phi$ , has a cycle, then we may unbrace any rhombus corresponding to an edge in the cycle, and obtain a bracing subgraph with one fewer edges.

**Proof:** Following the notation of the theorem, let

$$a, r_1, \dots, r_n, a$$

be a cycle in  $\Phi$ . By the transitivity lemma, we see that the existence of such a path implies that  $\angle ar_n$  is fixed. Thus we may remove the bracing of the rhombus  $(ar_n)$  without affecting the rigidity of the carpet.  $\square$

**DISTORTIONS UNDER THE WESTER-GAME**

To understand the possible distortions of a partially braced Penrose carpet, Chong Han programmed a

computer game which leads to a constructive proof of Wester's theorem.

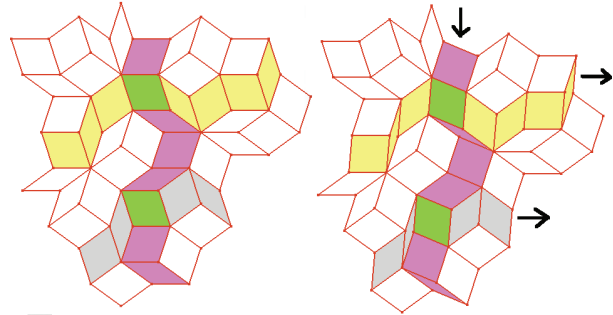


Fig. 04 The undistorted Penrose carpet on the left has two braced rhombi (green) in a vertical (purple) ribbon. A shear (vertical arrow) requires compensating shears along the horizontal crossing ribbons (yellow and grey). Crossing your eyes to fuse a pseudo-stereo image will identify those rhombi which remain fixed in this shift.

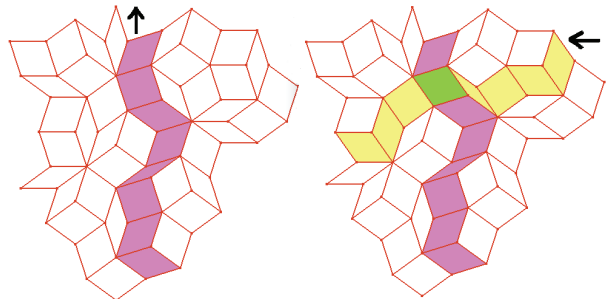


Fig. 05 Shifting the same Penrose carpet along the central vertical ribbon without bracings (left) merely shears it. We chose a negative angle this time. If the ribbon has one braced rhombus (green), then the crossing ribbon (yellow) had to shear left to keep the angles of the braced rhombus fixed.

As the game opens, the computer loads a carpet and computes its ribbons. The user selects/unselects the rhombi to be braced. The computer fixes the angles of the braced rhombi. The user checks the current rigidity of the carpet by choosing a ribbon,  $a$  and an angle  $\alpha$  along which to distort the carpet.

The distortion begins by rotating each edge in the direction of the ribbon  $a$  a small angle  $\alpha$ . If the ribbon has no braced rhombi in it, then this *shears* the ribbon, but one component of carpet complementary to the ribbon remains fixed, while the other component is translated rigidly by the shear. Even though only the ribbon shears, we shall call the entire distortion a *shear along  $a$  by angle  $\alpha$* .

If the selected ribbon,  $a$ , does have braced rhombi,  $(ab), (cd), \dots$  then the computer must restore each of them to their original shape, by shearing along the ribbons  $b, c, d, \dots$  which cross  $a$  by the same angle  $\alpha$ . Since these crossing ribbons may have additional braced rhombi, the algorithm must be recursive.

We shall call the resulting distortion a *shift*. This shift along the initial ribbon  $a$  by angle  $\alpha$  results in a cascade of shears and correcting shears to preserve the braced rhombs. The ribbons visited by the algorithm are marked to prevent an endless loop in the case that  $\Phi$  contains an edge cycle. If the algorithm, beginning at ribbon  $a$  visits all ribbons, then the carpet will be rotated rigidly by an angle  $\alpha$ . Otherwise, it will have distorted the carpet from its prior shape.

Thus, the user continues to brace rhombi and test for distortions. If a shift about *every* ribbon is a rigid rotation, then we have a rigidly braced carpet. Now consider the inverse problem. Is it perhaps possible to remove some of the bracings without affecting the rigidity? Bracing and unbracing rhombi in the carpet, either by trial and error, or by insight into Wester's theorem, the game with the computer will end by finding a minimally rigid bracing. Its associated Wester subgraph,  $\Phi$ , must be spanning, or there is a ribbon without braced rhombi in it. A shear along that ribbon distorts the carpet. On the other hand, unless  $\Phi$  is acyclic, there is some braced rhombus, corresponding to an edge in a cycle, which is redundant.

The last remaining question is why  $\Phi$  must be connected. Baglivo-Graver [2] state that a rigidly braced rectangular carpet must have a connected bracing graph  $\Phi$ . They outline a proof but leave the details to the reader. Our current argument, based on following the Wester game in greater detail, is too long for this paper. It will be presented at a later time.

### COUNTEREXAMPLES

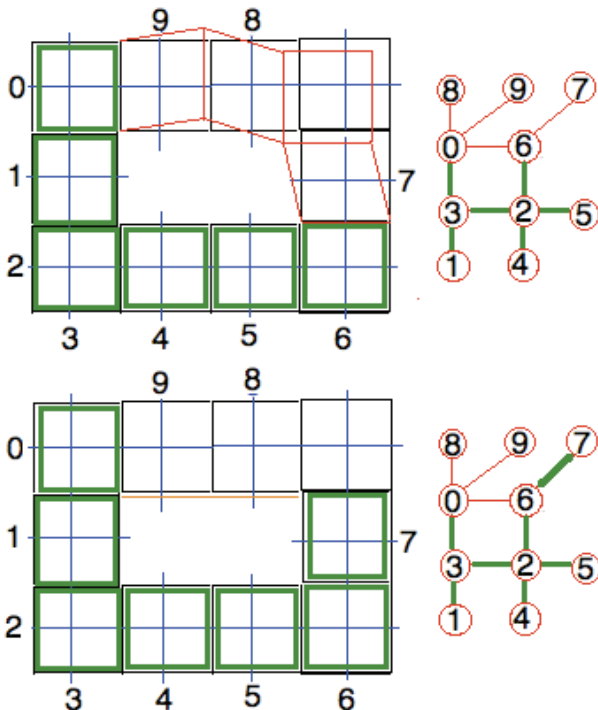


Fig. 05 Counterexamples for the Baglivo-Graver theorem for carpets of squares with a hole

The first example shows that Wester's theorem yields the optimal solution only when the carpet is *simply connected*, namely, when it has no "holes". We use a

carpet of squares. The Baglivo-Graver, [3, 4] theorem for a carpet of 3 by 4 squares predicts the need for  $6 = (3 + 4) - 1$  bracing plates. For example, along the left column and the bottom row. But, once we remove the central vertical rod, this bracing is no longer adequate to prevent a distortion of the carpet. The Wester graph, on the right is connected, and even a tree. But it is not spanning.

If we proceed and brace one more rhombus, (67), then the carpet becomes rigid, even though the Wester tree is still not spanning. The reason is that the shortest distance between (13) and (67) is exactly 2 side-lengths (red line). This forces, first (09) then (08) and finally (06) to all be horizontal squares.

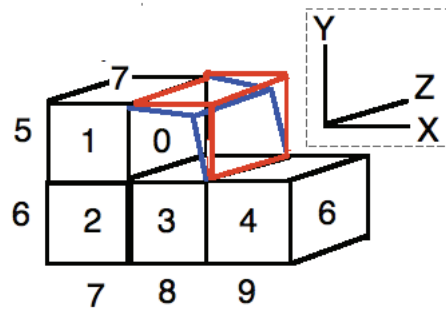


Fig. 06 Twisting distortion of a rod and pinion cube attached to a stepped wall of 4 rigidly braced cubes.

The second example shows that even the simplest generalization of our methods to 3D fails. Again we choose the Baglivo-Graver case for its simplicity. Consider a cubical framework with 5 cubes. We generalize the notion of ribbon to 3D as a maximal successions of contiguous rhombohedra which share a family of mutually parallel faces. To distinguish this from the 2D case, they shall be called *worms* instead of ribbons. We find two horizontal worms, 5,6, three vertical worms, 7,8,9 and five worms into the Z-direction, 0,1,2,3,4. Even if we brace all but one of the cubes, labeled

$$(157), (267), (368), (469)$$

for their crossing worms, the last red cube (058) distorts as its free edge slides forward into the negative Z-direction (blue). However, if we affix just one square plate to any of its four free faces, the framework becomes rigid.

The quasicrystalline 3D structures of interest to Robbin and others have additional geometrical structure which are needed for a solution to the bracing problem in 3D. Their generation by de Bruijn's [5] projection method permits the identification of two dimensional *curtains* of maximally contiguous rhombic faces, which generalize the notion of ribbons. Stabilizing curtains by bracing selected rhombic faces, restricting distortions to preserve parallelism, and discovering the right 3D version of the Wester tree is the subject of our future investigations.

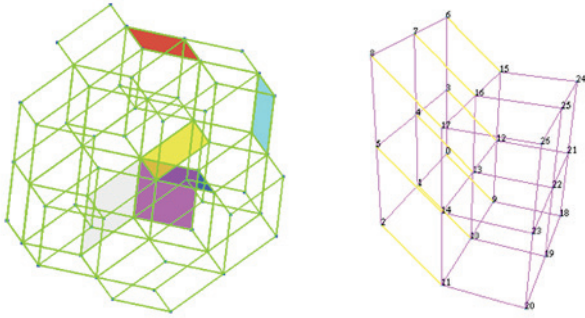


Fig. 07 Rhombohedral rod and pinion frameworks in 3D with too few bracing plates (left), and the kind of distortion which preserves rhombohedra (right).

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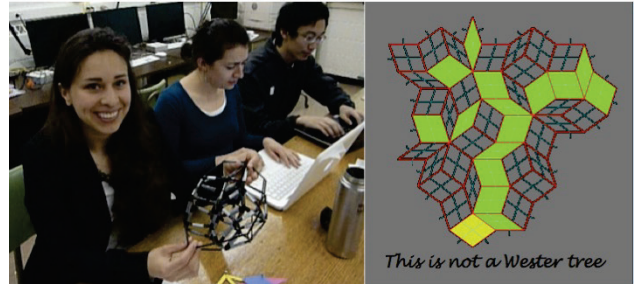


Fig. 08 The Illinois Geometry Lab team, Alex Burnley, Eliana Maria Duarte Gélvez, and Han Chong. Can you find the redundantly braced rhombi on the right?

#### References

- [1] Wester, T., "The Structural Morphology of Penrose and Quasicrystal Pattern" Part I, International Conference on Adaptable Building Structures, Adaptables 06, Eindhoven University of Technology, 2006. pp. 290-294.
- [2] Baglivo, J. Graver, J. "Incidence and Symmetry in Design and Architecture", Cambridge University Press. New York, 1983. pp 76-8  
[ISBN 0-521-23043-8](https://www.isbn-international.org/number/0-521-23043-8).
- [3] Robbin, T., "The Visual and Structural Properties of Quasicrystals", 2013 THIS VOLUME
- [4] Richeson, D. "An application of graph theory to architecture", <http://divisbyzero.com/2010/03/14/an-application-of-graph-theory-to-architecture>.
- [5] deBruijn, N.G. "Algebraic theory of Penrose's non-periodic tilings of the plane II", in Koninklijke Nederlandse Academie v. Wetenschappen, series A 84 1981. pp 37-66.