Single file made from all handwritten class files and annotation MA402Sp10.
Annotations for netMA402Sp11 in red edit boxes, such as this one. 4apr11.

W11 Calculations

Note Title 4/7/2010

Hristov's Proofs:

Review Moebor $\mathcal{M}$ = group of Moeb. Transfors.

8.1 $\{f \in Moeb \wedge f(z_i) = z_i \text{ for } i = 1,2,3\} \Rightarrow f = \iota$

8.2 $\{f, g \in \mathcal{M}_6 \wedge (\exists 3 z_i)(f(z_i) = g(z_i)\} \Rightarrow f = g$

8.3 $(\forall z_i, i = 0,1,2 \wedge w_i, i = 0,1,2)(\exists f \in \mathcal{M}_6)(f(z_i) = w_i)$

8.2 is a tricky but/ but easy application of 8.1

Because $f \in \mathcal{M} \Rightarrow f^{-1} \in \mathcal{M}$ } see earlier

Because $f, g \in \mathcal{M}_6 \Rightarrow f \circ g \in \mathcal{M}_6$ } notes

$\Rightarrow g^{-1} \circ f \in \mathcal{M} \wedge$ given $g^{-1} \circ f(z_i) = z_i \quad i = 1,2,3$

aside [Rem: $(A \Rightarrow B) \Rightarrow C(C \Rightarrow D) \wedge (C \Rightarrow A)$
another proof like this $C \wedge B \Rightarrow D$]

$\overset{8.1}{\Longrightarrow} g^{-1} \circ f = \iota \Rightarrow g = f$ done

Note: These lessons are for students who miss class on days we do "boardwork", for online students, and for those who want to check their class notes.

Also for corrections of errors or ambiguities

The example done in class below was less instructive than intended, mostly because it was not cooked well enough for the calculations to work out efficiently.

In a test don't expect to spend more than a minute on a pure calculation. If you've lost in a swamp of algebra you're probably doing the wrong thing.

ASIDE We showed that $\exists$ $\nearrow$ $\mathcal{PC}$ ... $= \frac{az+b}{cz+d}$, $a = a(z_0 z_1 z_2)$ etc

Recall $f(z) := CR(z; z_0 z_1 z_2) = \frac{z - z_0}{z - z_2} \cdot \frac{z_1 - z_2}{z_1 - z_0}$

$$(*) \begin{cases} f(z_0) = 0 \\ f(z_1) = 1 \\ f(z_2) = \infty \end{cases}$$ recall - substitute and simplify

$\longrightarrow$ to find $f(z_i) = w_i \quad i = 0, 1, 2$

I claim you need to solve $\Rightarrow$ for $w = f(z)$

$$CR(w, w_0 w_1 w_2) = \frac{w - w_0}{w - w_2} \cdot \frac{w_1 - w_2}{w_1 - w_0} = \frac{z - z_0}{z - z_2} \frac{z_1 - z_2}{z_1 - z_0} = CR(z, z_0 z_1 z_2)$$

why ? because of $*$

how ? Example $(1, i, -i) \xrightarrow{f} (1+i, 2, 1-i)$ with w_0, w_1, w_2

$$\rightarrow \frac{w - (1+i)}{w - (1-i)} \frac{2 - (1-i)}{2 - (1+i)} = \frac{z - 1}{z + i} \frac{i + i}{i - 1} \quad \Big| \quad \frac{2 - (1-i)}{2 - (1+i)} = \frac{1+i}{1-i}$$

$$\left(\quad \right) i = \left(\quad \right) \frac{2i}{i-1} \quad \Big| \quad = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1 + 2i + i^2}{1 - i^2}$$

I introduced in class. Thanks to Steve, Ethan, Elizabeth and Adrianna for catching faults

I forgot to say what I'm doing:

We solve = into form

$$w = \frac{\heartsuit z + \spadesuit}{\diamondsuit z + \clubsuit}$$

where

$\heartsuit = fcn(z_0 z_1 z_2 w_0 w_1 w_2)$

$$= \frac{2i}{2} = i$$

$$\frac{w-(1+i)}{w-(1-i)} = \frac{z-1}{z+1}\left(\frac{2}{i-1}\right)$$

side calculation

$$\frac{w-a}{w-b} = \frac{-az+a}{z+1}$$

$a = 1+i$
$b = 1-i$
$c = \frac{2}{i-1} = \frac{(i+1)2}{-2} = -(i+1)$

$$wz - az + w - a = -awz - ba + baz + w \quad = -a$$

$\overline{1231456789}9 \Longrightarrow$ Finish.

expect some interrelation to apply

$$\frac{w-a}{w-b} = \frac{z-c}{z-d} \Longrightarrow wz - aw \cdots = wz + \cdots$$

suppose the calc. were

$$w = \left(\frac{2i}{1-i}\right)z - \frac{i-1}{2}$$

check
$$w(1) = \frac{2i}{1-i} - \frac{i-1}{2} \neq 1+i$$

This is the 2nd calculation today. It does not seem to be in the current edition but is very important. It illustrates how to use the Anatomy Lesson (see Notes) constructively.

Theorem CR is a Moebius invariant

We need to show that

$f \in \mathcal{M} \Rightarrow f(CR(z_3, z_0, z_1, z_2)) \overset{*}{=} CR(f(z_3), f(z_0), f(z_1), f(z_2))$

Proof

Using the anatomy lesson, we check the for $\underbrace{\text{translation, rotation, dilation}}$ = simultitude — just as easy as each separately

first suppose $f(z) \overset{?}{=} az+b$. Substitute RHS int RHS (*)

$$\frac{(az_3+b)-(az_0+b)}{(az_3+b)-(az_2+b)} \; \frac{(az_1+b)-(az_2+b)}{(az_1+b)-(az_0+b)} = \text{lots of algebra}$$

The b's vanish in each numerator and denominator
Then the a's vanish by cancellation, leaving

just $\dfrac{z_3 - z_0}{z_3 - z_2} \; \dfrac{z_1 - z_2}{z_1 - z_0} = \text{LHS} (\ast)$ done!

Next, consider the case that $f(z) = \dfrac{1}{z}$

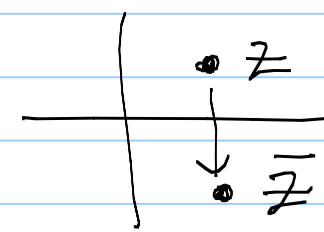
again, substitute in RHS($\ast$)

$$\frac{\frac{1}{z_3}-\frac{1}{z_0}}{\frac{1}{z_3}-\frac{1}{z_2}} \; \frac{\frac{1}{z_1}-\frac{1}{z_2}}{\frac{1}{z_1}-\frac{1}{z_0}} = \frac{(z_0-z_3)(z_2-z_1)}{(z_2-z_3)(z_0-z_1)} = \text{LHS } (*) \quad \text{done!}$$

Finall, also prove the invariance of <u>conjugation</u>

Compute $\overline{CR(z_3 z_0 z_1 z_2)}$ to show $= CR(\bar{z}_3 \bar{z}_0 \bar{z}_1 \bar{z}_2)$

recall $\overline{(x+iy)} = x - iy$



For this we use the <u>Algebraic Anatomy Lesson</u>

The CR is calculated by using only $+$, $-$, $\times$ and $\div$ of complex numbers. So, if we show that conjugation is invariant for <u>each</u> we are done.

Why is $\overline{z+w} = \bar{z}+\bar{w}$? Let $z = x+iy$ & $w = u+iv$

because

$$\text{RHS} = (x-iy)+(u-iv) = (x+u)-i(y+v)$$

$$= \overline{(x+u)+i(y+v)} = \overline{z+w}$$

Similarly for $-$, and more tediously for multiplication (do it!) but for division we use the trick of first showing

$$\overline{\left(\frac{1}{z}\right)} = \overline{\frac{\bar{z}}{|z|^2}} = \frac{\bar{\bar{z}}}{|z|^2} = \frac{z}{|z|^2} = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}$$

and writing $\frac{z}{w} = z\left(\frac{1}{w}\right)$.

End of Lesson W11

F11 Circlines

Note Title 4/9/2010

Hvidsten 8.1.2 in slow motion

After class annotations are in red.

8.6 $CR(a;b\,c\,d) \in \mathbb{R} \Rightarrow a\,b\,c\,d$ are cocirclinear — lie on the same circle or line

$\operatorname{Im} CR(\cdots) = 0 \Rightarrow a$ lies on circline (b,c,d)

for $f(z) := CR(z; z_0\, z_1, z_2) \overset{\exists a b c d}{=} \dfrac{az+b}{cz+d}$ (earlier calc)

proof that

$f(z) \in \mathbb{R} \Leftrightarrow \dfrac{az+b}{cz+d} \overset{*}{=} \dfrac{\bar a \bar z + \bar b}{\bar c \bar z + \bar d} \overset{\Delta}{=} \overline{\left(\dfrac{az+b}{cz+d}\right)}$

for Δ see M11

for * recall $\operatorname{Im}(w) = \dfrac{w - \bar w}{2i} = 0 \Leftrightarrow w = \bar w$

cross multiply *

$$\underbrace{(a\bar c - \bar a c)}_{\text{conj}} z\bar z + \overbrace{(a\bar d - \bar b c) z - (\bar a d - b \bar c)\bar z}^{\text{conj}} + \underbrace{(b\bar d - \bar b d)}_{\text{conj}} = 0$$

$$((a\bar d - \bar b c)z + b\bar d) - \overline{(\text{ its })} = 0$$

where $m := \bar{a}d - \bar{b}c$ and $k := b\bar{d} - \bar{b}d$

Case 1 $(a\bar{c} - \bar{a}c) = 0 \Longrightarrow \operatorname{Im}(mz + k) = 0$ is the equation of a st line by exercise.

exercise for F!!

So in Case 1 we have shown that $CR(z, z_0, z_1, z_2) \in \mathbb{R}$ iff z lies on the straight line with the equation "$\operatorname{Im}(mz + k) = 0$"

Case 2 $(a\bar{c} - \bar{a}c) \neq 0$ divide

$$z\bar{z} + \frac{(a\bar{d} - b\bar{c})}{a\bar{c} - \bar{a}c} z + \frac{(\bar{a}d - b\bar{c})}{-a\bar{c} + \bar{a}c}\bar{z} + \boxed{\frac{b\bar{d} - \bar{b}d}{a\bar{c} - \bar{a}c}} = 0$$

$$z\bar{z} + \gamma z + \bar{\gamma}\bar{z} + \delta = 0$$

where $\gamma = \dfrac{a\bar{d} - b\bar{c}}{a\bar{c} - \bar{a}c}$ and $\delta = \dfrac{b\bar{d} - \bar{b}d}{a\bar{c} - \bar{a}c}$ ◇

claim $\delta \in \mathbb{R} \Leftarrow \overline{\dfrac{b\bar{d} - \bar{b}d}{a\bar{c} - \bar{a}c}} = \dfrac{\bar{b}d - b\bar{d}}{\bar{a}c - a\bar{c}} = \dfrac{b\bar{d} - \bar{b}d}{a\bar{c} - \bar{a}c}$

unF.O.I.L(♡)

$$(z + \bar{\gamma})(\bar{z} + \gamma) - \bar{\gamma}\gamma + \delta = 0$$

$(w\bar{w} = |w|^2)$

$$|z - (-\bar{\gamma})|^2 = \bar{\gamma}\gamma - \delta = r^2$$

pos real = radius^2

equation of a circle centered at $-\bar{\gamma}$ radius $\sqrt{\gamma\bar{\gamma} - \delta}$

Details added after class (we ran out of time)
Continue with form ② of the equation.

LHS $(z+\bar{\gamma})(\bar{z}+\gamma) = (z-(-\bar{\gamma}))\overline{(z-(-\bar{\gamma}))} = |z-(-\bar{\gamma})|^2$

RHS $\gamma\bar{\gamma}-\delta$ first note that this is a real number because

$\gamma\bar{\gamma} = |\gamma|^2$ is and $\delta = \bar{\delta}$ see ◇. But we don't know its positive.

$$\gamma\bar{\gamma}-\delta = \frac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}c}\,\frac{\bar{a}d-b\bar{c}}{\bar{a}c-a\bar{c}} - \frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c} = \frac{\text{num}}{(a\bar{c}-\bar{a}c)(\bar{a}c-a\bar{c})} = \frac{\text{num}}{|a\bar{c}-\bar{a}c|^2}$$

num $= (a\bar{d}-\bar{b}c)(\bar{a}d-b\bar{c}) - (\bar{a}c-a\bar{c})(b\bar{d}-\bar{b}d)$ = FOILed again

$a\bar{d}\bar{a}d - a\bar{d}b\bar{c} - \bar{a}d\bar{b}c + \bar{b}cb\bar{c} + \bar{a}db\bar{c} - ad\bar{b}\bar{c} - \bar{a}\bar{d}bc + \bar{a}d\bar{b}c$

$= ad\bar{a}\bar{d} - \bar{a}\bar{d}bc - ad\bar{b}\bar{c} + bc\bar{b}\bar{c} = (ad-bc)(\bar{a}\bar{d}-\bar{b}\bar{c}) = |ad-bc|^2.$

Summarizing we have shown that the equation now has form

$$|z - p|^2 = r^2 \quad \text{where } p = -\bar{\gamma} \text{ and } r^2 = \frac{|ad - bc|^2}{|a\bar{c} - \bar{a}c|^2} > 0$$

which is the equation of a circle

$\mathrm{circ}(p, r)$ with center at k and radius r.

So, for Case 2 that $\bar{a}c \neq a\bar{c}$ we have $CR(z\ z_0 z_1 z_2) \in \mathbb{R}$

iff z lies on a circle

It must be $\mathrm{circ}(z_0 z_1 z_2)$

$$CR(z_0\ z_0 z_1 z_2) = \frac{z_0 - z_0}{z_0 - z_2}\,\frac{z_1 - z_2}{z_1 - z_0} = 0 \text{ is real}$$

$$CR(z_1, z_0, z_1, z_2) = \frac{z_1 - z_0}{z_1 - z_2} \; \frac{z_1 - z_2}{z_1 - z_0} = 1$$ is real and

$$CR(z_2, z_0, z_1, z_2) = \frac{z_2 - z_0}{z_2 - z_2} \; \frac{z_1 - z_2}{z_1 - z_0} = \infty$$ is also on the real axis

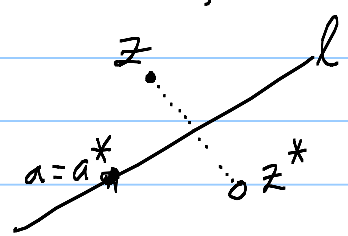
So z_0 z_1 z_2 are also on this circle. done!

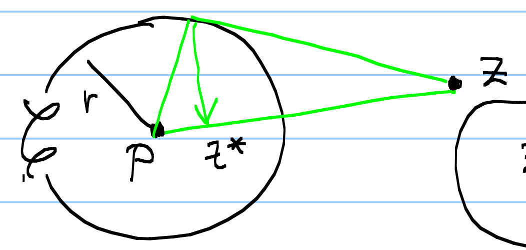
Note that circ(p, r) has eq'n $z\bar{z} - p\bar{z} - \bar{p}z + p\bar{p} - r^2 = 0$

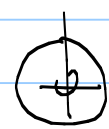
M12 symmetric points

Note Title 4/12/2010

Def: z & z^* are symmetric relative to a circline ℓ resp $\mathcal{C}$







Recall

$$\underbrace{|p - z^*|}\,|p - z| = r^2$$

$$z^* = p + \frac{r^2}{|z-p|}\,\frac{z-p}{|z-p|} = p + \frac{r^2}{|z-p|^2}(z-p)$$

by "vectors" and the vector equivalence.

$$= p + r^2 \frac{(z-p)}{(z-p)\overline{(z-p)}}$$

$$\therefore\ z = p + \frac{r^2}{\overline{(z-p)}}$$

8.7 z & z^* are symmetric rel a circle $\mathcal{C} = \mathrm{circ}(z_0 z_1, z_2)$

iff $CR(z^*\ z_0 z_1, z_2) = \overline{CR(z\ \ z_0\ z_1,\ z_2)}$

set $\heartsuit := \frac{r^2}{\overline{z-p}}$ for now

$$CR(z^* \; z_0 \; z_1 \; z_2) = CR(p+\heartsuit \; z_0 \; z_1 \; z_2) = \frac{p+\heartsuit - z_0}{p+\heartsuit - z_2} \, \frac{z_1 - z_2}{z_1 - z_0}$$

$$\overset{(1)}{=} CR(\heartsuit \quad z_0 - p \quad z_1 - p \quad z_2 - p)$$

recall $CR(f(z) f(z_0) f(z_1) f(z_2)) = CR(z \; z_0 \; z_1 \; z_2)$ if $f \in$ Moeb

$$\overset{2}{=} CR\left(\frac{r^2}{\overline{z-p}} \quad z_0 - p \quad z_1 - p \quad z_2 - p\right)$$

divide through by r2

$$\overset{3}{=} CR\left(\frac{1}{\overline{z-p}} \quad \frac{z_0 - p}{r^2} \quad \frac{z_1 - p}{r^2} \quad \frac{z_2 - p}{r^2}\right)$$

because $z \mapsto \frac{1}{z}$ is a Moeb transfo.

$$\overset{4}{=} CR\left(\overline{z-p} \quad \frac{r^2}{z_0 - p} \quad \frac{r^2}{z_1 - p} \quad \frac{r^2}{z_2 - p}\right)$$

by $\overline{CR(z \, z_0 \, z_1 \, z_2)} = cr(\bar{z} \; \bar{z}_0 \; \bar{z}_1 \; \bar{z}_2)$

$$\overset{5}{=} \overline{CR\left(z-p \quad \frac{r^2}{\overline{z_0 - p}} \quad \frac{r^2}{\overline{z_1 - p}} \quad \frac{r^2}{\overline{z_2 - p}}\right)} \overset{6}{=} \overline{CR(z-p \quad z_0^* \quad z_1^* \quad z_2^*)}$$

But each $z_i \in \text{circ}(z_0\, z_1\, z_2)$ so $z_i^* = z_i$

So

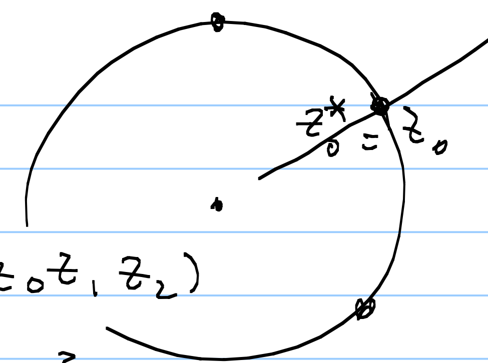
$$CR(z^*\ z_0\, z_1, z_2) \overset{7}{=} \overline{CR(z\ \ z_0\, z_1, z_2)}$$

if z^* and z are symmetric wrt $\text{circ}(z_0 z_1, z_2)$

conversely, if 7 holds, work back up

thru $\overset{6}{=}$ $\overset{5}{=}$ .... $\overset{1}{=}$ to show that $z^* = p + \dfrac{r^2}{\overline{z - p}}$

which says that z^* and z are symmetric for $\text{circ}(z_0\, z_1, z_2)$



W12 — conformal

Note Title 4/14/2010

Replaces Hvidsten 3.5.3 – you may but need not consult 3.5.3

Let $f(z) = \frac{az+b}{cz+d} = CR(z, z_0, z_1, z_2)$ etc

then $f'(z) = \frac{df(z)}{dz} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ if as in the calculus of real number functions

all of your favorite calculus works for complex numbers & functions

eg $f(z) = az + b$ similarities to calculate $f'(z)$ do:

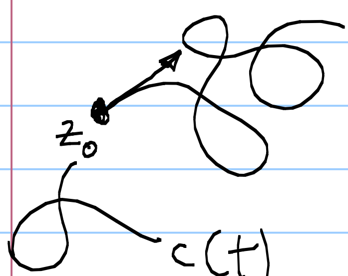
(1) $\frac{a(z+h)+b - az - b}{h} = \frac{ah}{h} = a \to a$

(2) $f(z) = z^2$

$\frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h$

$f(z) = \frac{az+b}{cz+d}$ $\quad f'(z)$ everywhere

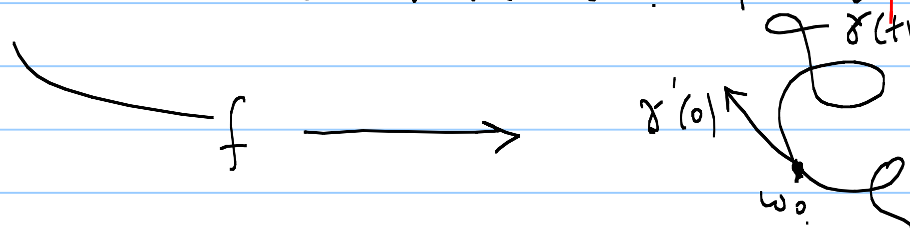
What geometrical meaning does $f'(z)$ have
in particular - what happens to
directions, velocities, vectors ....



$c(t) \in \mathbb{C}$, $t \in \mathbb{R}$, $c(0) = z_0$

$c'(t)$ = velocity vector of $c(t)$

at $t = 0$, $c'(t)$
is the velocity of $c(t)$
hence $c'(0)$ is the velocity at z_0



etc

as $h \to 0$ $\quad Zz = \frac{dz^2}{dz}$

Using quotient rule

$$f'(z) = \frac{a(cz+d)-(az+b)c}{(cz+d)^2}$$

which can be evaluated

e.g. $f'(0) = \frac{ad-bc}{d^2}$

but observe that
$f'(z)$ is not Möbius

Continue F12

Now look at the curve after a transformation and call the image curve

$\gamma(t) = f(c(t))$ then $f(c(0)) = f(z_0)$ call it w_0

and by the chainrule (which is part of the calculus)

$\gamma'(t) = f'(c(t))c'(t) \Rightarrow \gamma'(0) = f'(z_0)\,c'(0)$

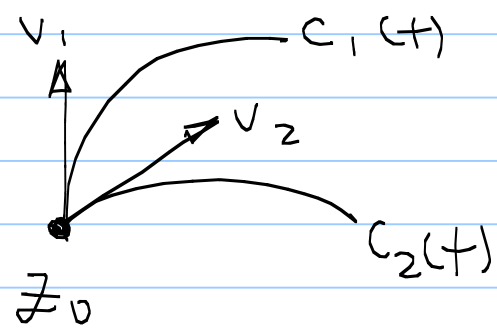
similarity velocity vector

Suppose $0 \neq f'(z_0) = re^{i\theta}$ in polar form, then

$\gamma'(0) = \underline{re^{i\theta}}\,c'(0)$ i.e. under f, $c'(0)$ gets rotated by θ

(it is also dilated by r – but we don't need that here)

<u>Theorem</u> Given two direction vectors v_1, v_2 at z_0 tangent to curves $c_1(t), c_2(t)$ at z_0, i.e.



$$c_1(0) = c_2(0) = z_0 \wedge c_i'(0) = v_i \quad i = 1, 2$$

and $f(z)$ a differentiable function (for example $f(z) = \frac{az+b}{cz+d}$ or $f(z) = \bar{z}$)

and $\gamma_i = f \circ c_i$ then $\angle(\gamma_1'(0), \gamma_2'(0)) = \angle(v_1, v_2)$.

Proof

Substitute and note that

$\angle \gamma_1'(0), \gamma_2'(0) = \angle re^{i\theta}c_1'(0), re^{i\theta}c_2'(0)$ where $f'(z_0) = re^{i\theta}$

$= \angle c_1'(0)\, c_2'(0)$ because

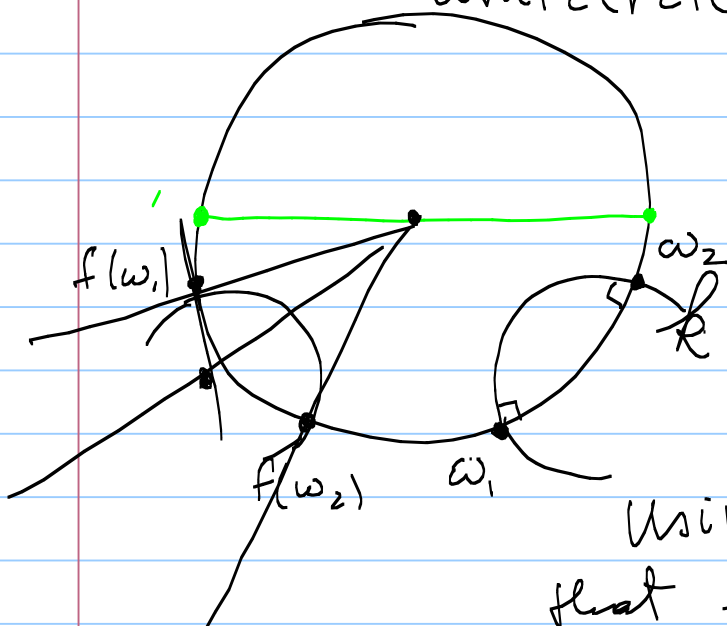
dilations change no angles, and both vectors are rotated by the same angle θ.

done

Applications

$$\left\{ f \;\middle|\; f(z) = e^{i\theta}\,\frac{z-\alpha}{\bar{\alpha}z - 1} \right\} = \text{hyperbolic isometries of the Poincaré plane model of non-Euclid. geometry}$$

for short "Poincaré Isometries"



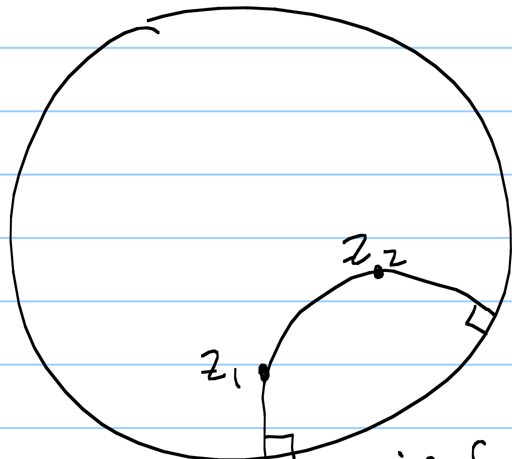
(1) $|z| = 1 \implies |f(z)| = 1$

(2) $|z| < 1 \implies |f(z)| < 1$

where does f take k (a PLine)

Using conformality and (1) we know that $f(k)$ is also a Poincare Line

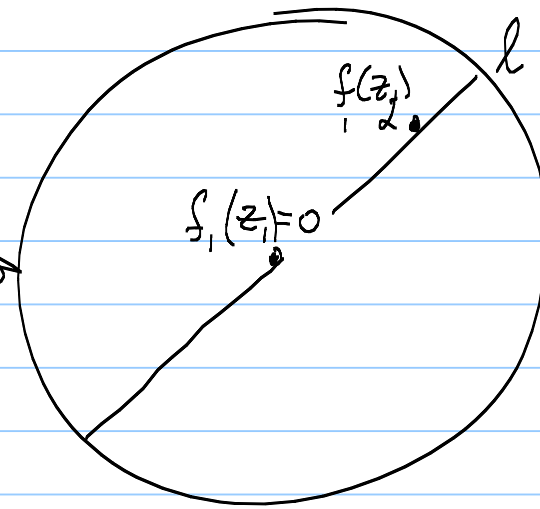
$f \in$ P-isometry $\implies$ PLines go to PLines



Euclid I

Given $|z_1|$ $|z_2| < 1$

Find
$f_1 \in$ PIsom



$f_1(z_1) = 0$

i.e. Solve $e^{i\theta}\dfrac{z_1-\alpha}{\bar{\alpha} z_1 - 1} = 0$ for α

$z_1 = \alpha$

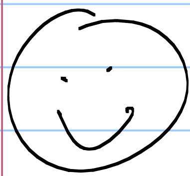
So $f_1(z) = \dfrac{z - z_1}{\bar{z}_1 z - 1}$ (i.e. $f = T_{-z_1}(z)$)

$f_1(z_2)$ goes wherever the diameter l thru $f_1(z_2)$ is a P line

Hence $f_1^{-1}(\ell)$ is the P-line thru z_1, z_2

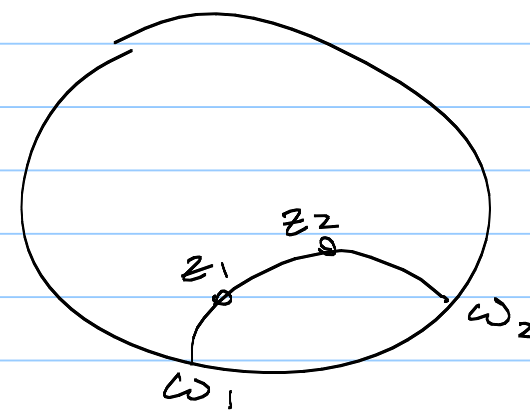
Recall $f(z) = \dfrac{z-\alpha}{\bar{\alpha} z - 1}$ then $f^{-1}(z)$

$T_{-\alpha}(z) \Rightarrow T^{-1}_{-\alpha}(z) = T_{\alpha}(z)$



$w = \dfrac{z-\alpha}{\bar{\alpha} z - 1}$ solve for z

$f^{-1}(z) = \dfrac{z+\alpha}{\bar{\alpha} z + 1}$

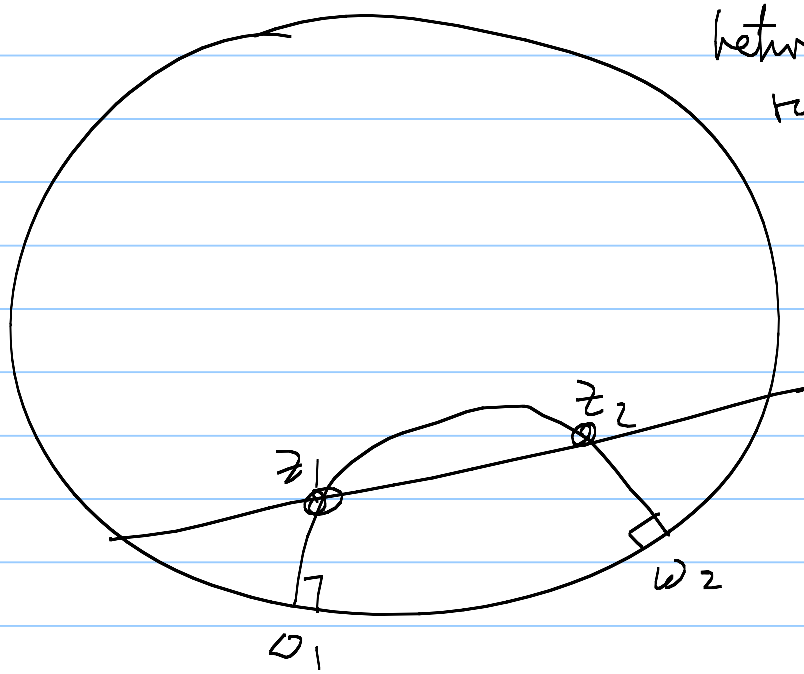


Payoff

$|\ln CR(z_1, \omega_1, z_2, \omega_2)| = \mathrm{dist}_{\underline{P}}(z_1, z_2)$

is invariant under all $f \in \mathrm{isom}_{\underline{P}}$.

Preview



What is the relation between the Poincaré ruler thru given z_1, z_2 and the Klein ruler?

$\text{dist}_K(z_1, z_2)$

M13 – Symmetry

Note Title 4/19/2010

Motivation: The quiz on F13 covers weeks 12 and 11. So this week we'll discuss issues that reviews this work. Today we compare the geometric and algebraic methods for exploring the consequences of symmetry.

In this transcript of the lecture, green and red are commentaries. The geometrical approach, based on Thales' theory of similar right Δs, led to the formula $|z-p||z^*-p| = r^2$ for points symmetric relative to a circle centered at p of radius r.

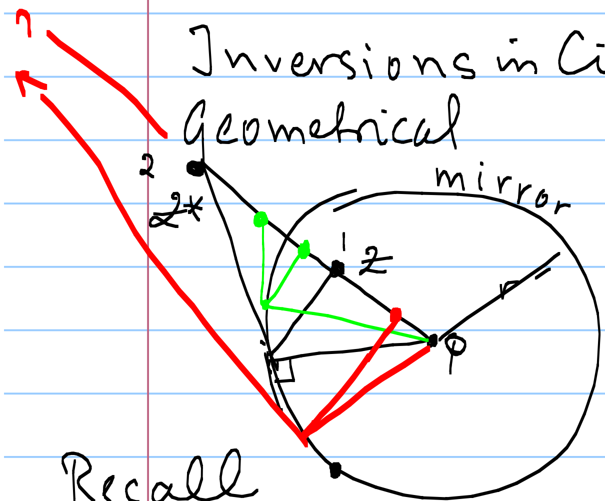
From this we can "read" the figures for the first 4 properties but (5) is too difficult to prove geometrically.

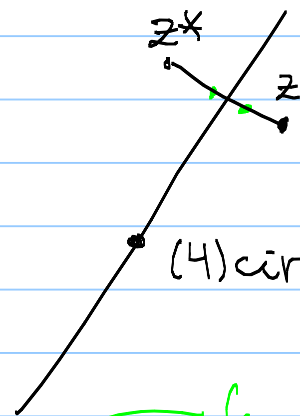
The first colored figure suggests that $\overbrace{\lim_{z\to\text{mirror}} z^* = z}^{\text{green}}$ & $\overbrace{\lim_{z\to 0} z^* = \infty}^{\text{red}}$

Note Title 4/19/2010

Inversions in Circles = Symmetries = Reflections in circlines

Geometrical





(1) $z^{**} = z$

(2) $z \in \text{mirror} \Rightarrow z^* = z$

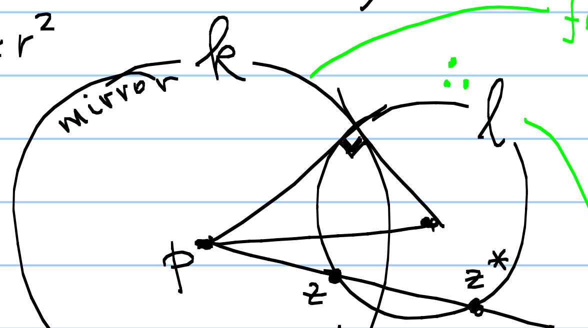
(3) $c = \text{center} \Rightarrow c^* = \infty$

(4) circlines $\ell \perp k \Rightarrow k \perp \ell$

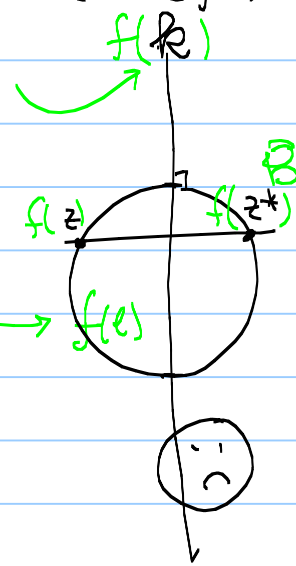
(5) If $\ell \perp k$ then $\ell^* = \ell$.

Recall

$|z-p||z^*-p| = r^2$



$f \in$ moeb



geometrically this is nontrivial

algebraically it is trivial

Möbius Transformations

Big Theorem

If $f(z) \in$ Möb then

(a) f takes cirlines to circlines

(b) symmetric points to symmetric points

Comment after class on the previous panel

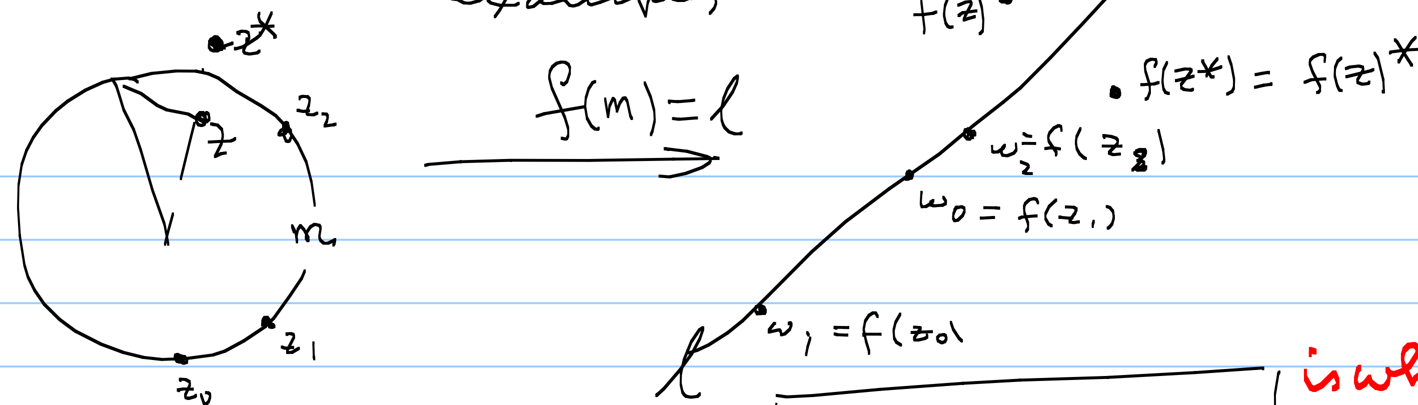
The Big Symmetry Theorem (BST) is useful to prove (5) as follows.

Given circle k and circle $l \perp k$ as on the left. Choose some $f \in$ möb with $f(p) = \infty$, then $f(k)$ is a straight line, and $f(l)$ is a 'circline' and $f(l)$ is still $\perp$ to $f(k)$.

It can't look like . Now it is clear that $f(l)$ is symmetric in the classical sense. The BST also says that if z, z^* are symmetric rel to k, $f(z)$ and $f(z^*)$ are symmetric rel to $f(k)$. Since $f(l)$ is symmetric across the line $f(k)$, its preimage l is symmetric across k.

The next panel just illustrates what I just said. Since the meaning of the $*$ notation depends on a context, let's replace it by placing the <u>name</u> of the mirror of symmetry.

examples



if we write $z^* = z^m = S_m(z)$ then $\boxed{f(z^m) = f(z)^{f(m)}}$ is what we have proved

i.e. for $w := f(z)$ and $f(k) := h$ then $f(z^m) = w^h$

Comment: Given circline k and circline $l \perp k$ (4 figures)
then k is symmetric rel. to l

Pf: Send k to straight line $f(k)$ by some Moeb.
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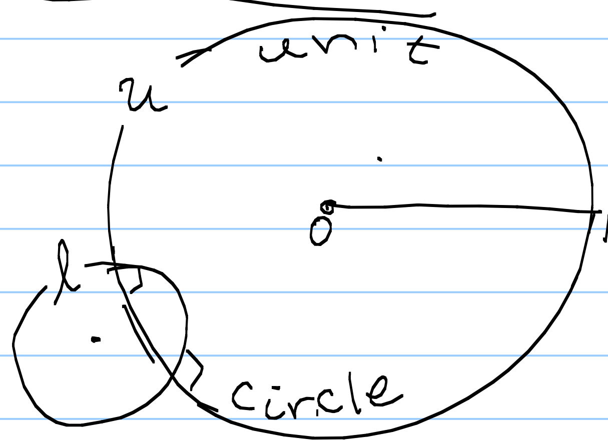
Another example



What is the equation of a circle $\perp$ Unit Circle?

Recall

$|z+\mu|^2 = r^2$ is the equation of circ$(-\mu, r)$

i.e. centered at $-\mu$ and radius r.

Calculate

$(z+\mu)(\bar{z}+\bar{\mu}) = r^2$

$z\bar{z} + \mu\bar{z} + \bar{\mu}z + |\mu|^2 = r^2$

$$(*) \quad z\bar{z} + \mu\bar{z} + \bar{\mu}z + (|\mu|^2 - r^2) = 0$$

⚠ If the circle is symmetric w.r.t. the unit circle then its equation must be $\boxed{z\bar{z} + \mu\bar{z} + \bar{\mu}z + 1 = 0}$

pf: Subst $\frac{1}{\bar{z}}$ into $(*)$ $\quad \frac{1}{\bar{z}} \cdot \frac{1}{z} + \mu \frac{1}{z} + \bar{\mu}\frac{1}{\bar{z}} + (\delta) = 0$

multiply $z\bar{z}$ $\quad 1 + \mu\bar{z} + \bar{\mu}z + (\delta)z\bar{z} = 0$

which is the original eqn iff $\delta = 1$

Elaboration:

The circle in question is $\{z \mid z\bar{z} + \mu\bar{z} + \bar{\mu}z + \delta = 0\} = \ell$ (where $\delta := |\mu|^2 - r^2$) and we say it is $\perp u$.

So, if $z \in \ell$ then $z^* \in \ell$ as well. So it must satisfy the equation. Since $z^* = z^u = \dfrac{1}{\bar{z}}$, substituting we find

$1 + \mu\bar{z} + \bar{\mu}z + \delta z\bar{z} = 0$ for every $z \in \ell$. The only way the two equations can be for the same circle is that $\delta = 1$.

Conversely, the circle $z\bar{z} + \mu\bar{z} + \bar{\mu}z + 1 = 0$ is $\perp u$ because the equation is invariant under the symmetry $z \to z^* = 1/\bar{z}$.

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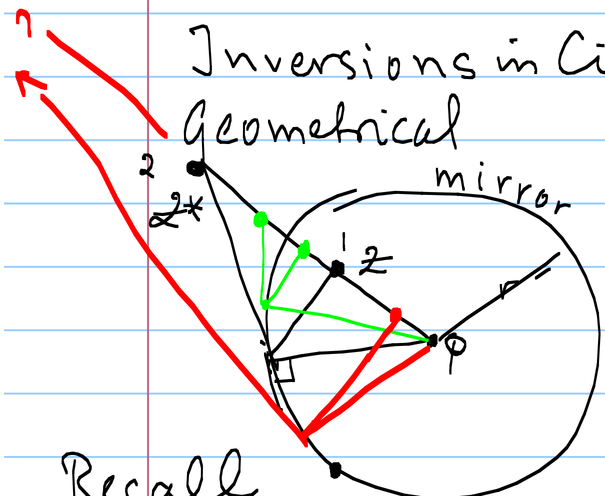

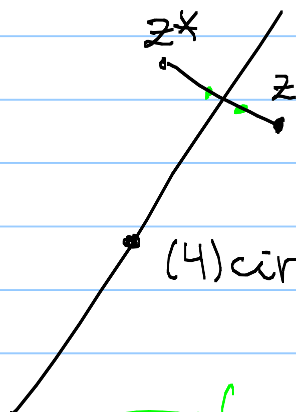

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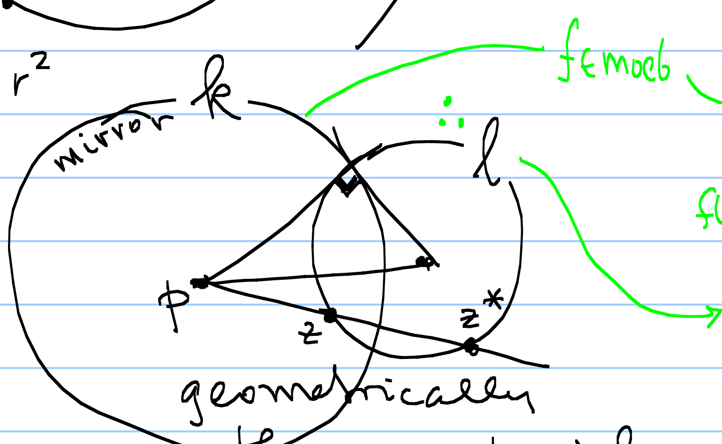

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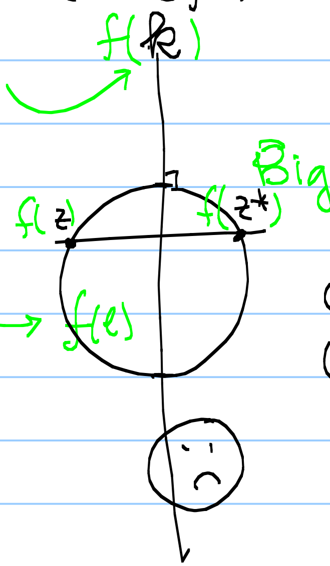

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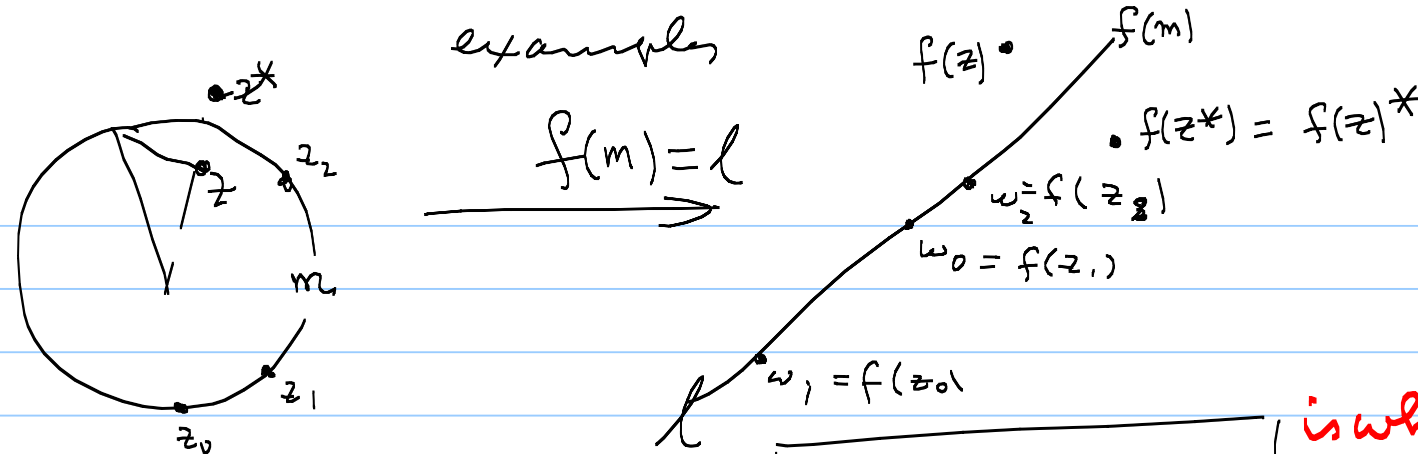

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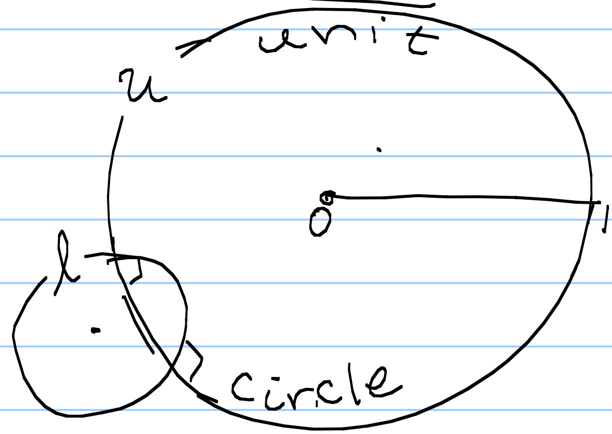

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