Axiomatic Systems for Geometry*

George Francis†

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1 Basic Concepts

An axiomatic system contains a set of *primitives* and *axioms*. The primitives are object names, but the objects they name are left undefined. This is why the primitives are also called *undefined terms*. The axioms are sentences that make assertions about the primitives. Such assertions are not provided with any justification, they are neither true nor false. However, every subsequent assertion about the primitives, called a *theorem*, must be a rigorously logical consequence of the axioms and previously proved theorems. There are also formal *definitions* in an axiomatic system, but these serve only to simplify things. They establish new object names for complex combinations of primitives and previously defined terms. This kind of definition does not impart any 'meaning', not yet, anyway.

Adaptation to the current course is in the margins.

See Lesson A6 for an elaboration.

If, however, a definite meaning is assigned to the primitives of the axiomatic system, called an *interpretation*, then the theorems become meaningful assertions which might be true or false. If for a given interpretation all the axioms are true, then everything asserted by the theorems also becomes true. Such an interpretation is called a *model* for the axiomatic system.

In common speech, 'model' is often used to mean an example of a class of things. In geometry, a model of an axiomatic system is an interpretation of its primitives for

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†Prof. George K. Francis, Mathematics Department, University of Illinois, 1409 W. Green St., Urbana, IL, 61801.

which its axioms are true. Since a contradiction can never be true, an axiom system in which a contradiction can be logically deduced from the axioms has no model. Such an axiom system is called *inconsistent*. On the other hand, if an abstract axiom system does have a model, then it must be consistent because each axiom is true, each theorem is a logical consequence of the axioms, and hence it is true, and a contradiction cannot be true.

Finally, an axiom system might have more than one model. If two models of the same axiom system can be shown to be structurally equivalent, they are said to be *isomorphic*. If all models of an axiom system are isomorphic then the axiom system is said to be *categorical*. Thus for a categorical axiom system one may speak of *the* model; the one and only interpretation in which its theorems are all true.

Hvidsten, in his textbook "Geometry with Geometry Explorer", McGrawHill, 2004, devotes a large part of Chapter 1 giving examples for these concepts using finite geometries. Earlier editions of MA402 treats this subject as well. In this edition of the course, we regrettably dispense with it in the interest of having more time on other topics.

All of these qualities: truth, logical necessity, consistency, uniqueness were tacitly believed to be the hallmark of classical Euclidean geometry. At the start of the 19th century, a scant 200 years ago, philosophers and theologians, physicists and mathematicians were all persuaded that Euclidean geometry was absolutely the one and only way to think about space, and therefore it was the job of geometers to develop their science in such a way as to demonstrate this necessary truth. By the end of the century, this belief had been thoroughly discredited and abandoned by all mathematicians.[1]

The main theme of our course concerns the evolution of this idea, and its replacement by the much richer, far more illuminating, post-Euclidean geometry of today. It is about a method of thought, called the *axiomatic method.* Although at one time this method may have developed merely from a practical need to verify the rules obtained from the careful observation of physical experiments, this changed with the Greek philosophers. The axiomatic method has formed the basis of geometry, and later all of mathematics, for nearly twenty-five hundred years. It survived a crisis with the birth of non-Euclidean geometry, and remains today one of the most distinguished achievements of the human mind.

As we noted earlier, the transition of geometry from *inductive inference* to *deductive reasoning* resulted in the development of axiomatic systems. Next, we look at four axiom systems for Euclidean geometry, and close by constructing a model for one of them.

[1]Curiously, it persists even today among some irresponsible, but influential amateurs. In her column, "Ask Marilyn" in Parade Magazine, November 21, 1993, the world's most intelligent woman [sic] rejects Hyperbolic Geometry on nonsensical grounds, see `http://en.wikipedia.org/wiki/Marilyn_vos_Savant`.

2 Euclid's Postulates:

Earlier, we referred to the basic assumptions as 'axioms'. Euclid divided these assumptions into two categories — postulates and axioms. The assumptions that were directly related to geometry, he called *postulates*. Those more related to common sense and logic he called *axioms*. Although modern geometry no longer makes this distinction, we shall continue the ancient custom and refer to axioms for geometry also as postulates.

Here is a paraphrase[2] of the way Euclid expressed himself.

Let the following be postulated:

MA402 lessons A2 and E1 ff

Postulate 1: To draw a straight line from any point to any point.

Postulate 2: To produce a finite straight line continuously in a straight line.

Postulate 3: To describe a circle with any center and distance.

Postulate 4: That all right angles are equal to one another.

Postulate 5: That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Note that the wording suggests construction. Euclid assumed things that he felt were too obvious to justify further. This caused his axiomatic system to be logically incomplete. Consequently, other axiomatic systems were devised in an attempt to fill in the gaps. We shall consider three of these, due to David Hilbert, (1899), George Birkhoff, (1932) and the School Mathematics Study Group (SMSG), a committee that began the reform of high school geometry in the 1960s.

[2]Thomas L. Heath, "The Thirteen Books of Euclid's Elements", Cambridge, 1908.

3 Hilbert's Postulates:

In the late 19th century began the critical examinations into the foundations of geometry. It was around this time that David Hilbert(1862 - 1943) introduced his axiomatic system. The primitives in Hilbert's system are the sets of *points*, *lines*, and *planes* and relations, such as

The following sections may be skimmed with profit now, but will not be easy to follow until additional lessons on Euclid's Elements and the his Parallel Postulate have been studies.

incidence:	as in 'a point A is on line ℓ'
order:	as in 'C lies between points A and B'
congruence:	as in ' line segments $AB \cong A'B'$'

An example of a formal definition would be that of a *line segment* AB as the set of points C between A and B. He partitioned his axioms into five groups; axioms of connection,order, parallels, congruence and continuity.[3] Hilbert's axiom system is important for the following two reasons. It is generally recognized as a flawless version of what Euclid had in mind to begin with. It is purely geometrical, in that nothing is postulated concerning numbers and arithmetic. Indeed, it is possible to model formal arithmetic inside Hilbert's axiomatic system.

We wish to show how Euclidean geometry can be modelled inside arithmetic.[4] For this purpose, we want the shortest possible list of primitives and postulates, for then, we have less to check. Birkhoff meets this requirement.

4 Birkhoff's postulates

The primitives here are the set of *points*, a system of subsets of points called *lines*, and two real-valued functions, 'distance' and 'angle'. That is, for any pair of points, the distance $d(A, B)$ is a positive real number. For any ordered triple of points A, Q, B,

These concepts are treated in greate detail in the module on Cartesian Geometry, Lesson C1. And also in Hvidsten *op. cit.* Chapter 3

[3] cf. Wallace and West, "Roads to Geometry", Pearson 2003, Chapter 2 for a more detailed discussion of Hilbert's axioms.

[4] The historical significance of these two exercises in building models of formal systems is the irrefutable demonstration that geometry and arithmetic are equi-consistent. That means, if you believe the one to be without contradiction, then you are obliged to accept the other also, and vice-versa. Hilbert's program for a proof that one, and hence both of them are consistent came to naught with Gödel's Theorem. According to this theorem, any formal system sufficiently rich to include arithmetic, for example Euclidean geometry based on Hilbert's axioms, contains true but unprovable theorems.

the real number $m\angle AQB$ is well defined[5] modulo 2π.

Euclid's Postulate: A pair of points is contained by one and only one line.

Ruler Postulate: For each line there is a 1:1 correspondence between its points and the real numbers, in such a way that if A corresponds to the real number t_A and B corresponds to t_B then

$$d(A, B) = |t_B - t_A|$$

.

Protractor Postulate: For each point Q, there is a 1:1 correspondence of its rays[6] and the real numbers[7] modulo 2π, in such a way that if ray r corresponds to the circular number ω_r and ray s to ω_s then

$$m\angle RQS = \omega_s - \omega_r (mod 2\pi)$$

where R is a second point on r and S on s .

simSAS Postulate: If $m\angle PQR = m\angle P'Q'R'$ and $d(PQ) : d(P'Q') = d(QR) : d(Q'R') = k$ then the other four angles are pairwise equal, and the remaining side pairs have the same ratio.

One says that such triangles are *similar*, $\triangle PQR \sim \triangle P'Q'R'$ with *scaling factor* k. Of course, for $k = 1$, $\triangle PQR \cong \triangle P'Q'R'$.

5 The SMSG Postulates

There are 22 of these,[8] and they combine the flavor of Hilbert and Birkhoff. With Birkhoff, rulers and protractors are postulated, under the valid impression that children already know how to deal with real numbers by the time they study geometry.

[5]To distinguish the figure $\angle AQB$, which we call an 'angle', the number $m\angle AQB$ is called the *angular measure* of the angle. Moreover, two real numbers that differ by a multiple of 2π measure the same angle.

[6]Note that once we can apply a ruler to a line, we can identify one of the two half-lines, or rays, at a point Q as those points P on the line for which $t_P > t_Q$.

[7]We might call these the *circular numbers* because they lie on the *number circle*, just as one speaks of the real numbers lying on *number line*.

[8]Cf. Appendix of Wallace and West, *op.cit.*

There are many postulates so that proofs of interesting theorems can be constructed without the tedium of proving hundreds of lemmas first. Of course, unlike Birkhoff's foursome, the SMSG postulates are redundant, in that some postulates can be logically derived from others. The pedagogical wisdom and usefulness of the SMSG axiom system is a matter of some debate among educators.

6 A Cartesian Model of Euclidean Geometry

Hvidsten 3.6. Class lesson C2

We next give an example of an axiomatic system and a model for it. For this purpose we choose a very familiar area of mathematics in which to interpret the primitives and to test the truth of the axioms. We all know analytic plane geometry from high school, also known as *Cartesian geometry*. Birkhoff's four postulates for Euclidean geometry appear compact enough for us not to lose our way.

We interpret the *points* $A, B, C...$ as ordered pairs, (x, y), of real numbers. *Lines* shall be solution sets to linear equations of the form $ax + by + c = 0$. A point (p, q) is *incident* to the line $ax + by + c = 0$ if it satisfies the equation, i.e. if $ap + bq + c = 0$ is true. Remember that the distance between two points and the angle measure are also primitives and need an interpretation. We shall do that later.

With just this much we can already attempt to verify the first postulate which asserts the existence and uniqueness of a line through two given points. You could do this yourself by deriving the formula for the line through two points $(x_0, y_0), (x_1, y_1)$ in any of the many ways you learned to do this in high school. Here we do this by solving this system of two linear equations for the as yet unknown parameters a, b, c :

$$\begin{array}{rcl} ax_0 + by_0 + c & = & 0 \\ ax_1 + by_1 + c & = & 0 \\ a(x_1 - x_0) + b(y_1 - y_0) & = & 0 \end{array}$$

The third equation eliminates c for the moment; we can recover it as soon as we know a, b, for example thus:

$$c = -ax_0 - by_0.$$

One plausible choice for a,b would be

$$a = -(y_1 - y_0),\ b = (x_1 - x_0)$$

because it fits the third equation and yields

$$c = x_0y_1 - x_1y_0 = \begin{vmatrix} x_0 & y_0 \\ x_1 & y_1 \end{vmatrix}$$

While this shows that both points lie on some line, it does not demonstrate the uniqueness of this line. Indeed, our interpretation is incomplete. If we really want the first postulate to hold we must agree that the same line may have more than one equation, provided the same set of points is the solution set for each. We therefore amend our interpretation of a line by stressing that

$$\begin{aligned} a_0x + b_0y + c_0 = 0 \\ a_1x + b_1y + c_1 = 0 \end{aligned}$$

define the same line provided the parameters are proportional:

$$a_0 : a_1 = b_0 : b_1 = c_0 : c_1.$$

The distance function $d(A, B)$ Birkhoff had in mind is, of course, the Euclidean distance as derived from the Pythagorean theorem:

For $A = (x_0, y_0), B = (x_1, y_1)$,

$$d(A, B) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

We are now ready to verify Birkhoff's ruler postulate in a particularly useful fashion. First we give our two arbitrary points more mnemonic names: $Q = (x_0, y_0), I = (x_1, y_1)$. Now there is a canonical way of labelling all other points on the line QI with real numbers t in several useful ways as follows:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} (1 - t) + t \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

In vector notation this might be written as

$$P_t = Q(1 - t) + tI = Q + t(I - Q).$$

Notice that $P_0 = Q$ and $P_1 = I$, and that the points on the segment QI are given by the set

$$\{P_t | 0 < t < 1\}.$$

There is still something to prove here, namely that the Euclidean distance is in fact measured by our ruler. Once again we were too hasty in ruling lines. For the Euclidean distance

$$d(Q, I) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

which need not equal the parametric distance, which is 1. We may, however, rescale our ruler by a unit $u = d(Q, I)$, to yield another parameter, $s = tu$, for which the points P_s on the line are given by $P_s = Q + sU$, where U is the unit vector, $U = (I - Q)/d(I, Q)$, in the direction if I from Q . This way, I is the correct distance, $d(I, Q)$, away from Q on this ruler for the line.

For the remaining pair of Birkhoff's postulates we need a protractor, i.e. a device for measuring angles. The simplest way of doing this in our model is to recall the definition of the dot product of two vectors and interpret:

$$m\angle AQB = \arccos(\frac{A - Q}{d(A, Q)} \cdot \frac{B - Q}{d(B, Q)}).$$

Birkhoff's axiom system achieved its remarkable economy by postulating what turns out to be the quintessential property of Euclidean geometry. What distinguishes it from non-Euclidean geometry are the properties of *geometric similarity*. Two shapes are similar if they differ only in scale. Birkhoff postulates that two triangles with a similar corner, are wholly similar. By a corner we mean, of course, a vertex and its adjacent edges. If the proportionality factor is 1, then this postulate says that two triangles are congruent as soon as they have one congruent corner.

We shall verify the simSAS postulate, which makes an assertion about two triangles, by carefully measuring one triangle. Just as today we exchange goods by means of their price, instead of bartering items for each other, so modern geometry compares shapes by comparing their measurements.

Given a triangle $\triangle ABC$, vital statistics consists of six numbers, the three angles and sides,

$$\begin{gathered}\alpha = m\angle A \\ \beta = m\angle B \\ \gamma = m\angle C \\ a = d(B, C) \\ b = d(C, A) \\ c = d(A, B).\end{gathered}$$

The law of cosines, which generalizes the Pythagorean theorem to arbitrary triangles by resolving the square of a side in terms of the opposite corner:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

allows us to measure c in terms of the measures of two sides and the included angle.

7 Problems

These problems are assigned for submission elsewhere. As you study this lesson and as you solve these problems, enter their solutions into your Journal for future reference.

Problem 1:

Above we have given the parametric equations for a line. Recall how to eliminate the paramater t to obtain the non-parametric equation of a line you may be more familiar with.

Problem 2:

To demonstrate that you understood our verification of Birkhoff's Ruler Axiom, see if you can do likewise for his Protractor Axiom. In other words, you first define the angle measure of every angle as some circular number between 0 and 2π. Recall that was done in terms of a unit circle centered at the vertex of the angle. The angle measure is the length of the arc between the two points the legs of the angle cross the circle.

Next, show that parametrization of this unit circle by the cosine and sine of a real number establishes a bijection to the circular reals of the angle from the from the positive x-axis. There remains to prove that any angle situated at that origin is measured by the difference of the parameters.

As you complete this non-trivial exercise, you will discover that you need to use a well known trig identity.

Problem 3:

To finish the proof that Birkhoff's simSAS axiom holds in Cartesian geometry, you should first use vector algebra and the definition of the dot product to verify the law of cosines. Hint: Multiply out

$$C^2 = (B - A)^2 = ((B - C) - (A - C))^2.$$

Thus, knowing a, b and γ, we calculate c. If a and b are stretched, or shrunk by the same factor, so is c, provided γ remains the same.

Problem 4:

Continuing. Thus, knowing a, b and γ, we calculate c. If a and b are stretched, or

shrunk by the same factor, so is c, provided γ remains the same.

Apply the law of cosines to the other two sides to calculate α and β as functions of a, b and γ.

Problem 5:

A generalization of Euclid's proof of the Pythagorean theorem leads to another proof of the law of cosines. Label an arbitrary acute triangle in the standard way. Construct squares on two of its sides, say b and c. Extending the altitudes from C and B partitions the squares into rectangles

$$\begin{aligned} b^2 &= bb_1 + bb_2 \\ c^2 &= cc_1 + cc_2 \end{aligned}$$

Euclid's argument (do it!) proves that $cc_1 = bb_2$

Now drop the third altitude from A. Of course (can you prove this?) it passes through the same point where the first two altitudes intersected,[9] and partitions the third square into two rectangles.

$$\begin{aligned} aa_2 = &\quad bb_2 = ab\cos C \\ c^2 = &\quad cc_1 + cc_2 \\ = &\quad bb_2 + aa_1 \\ = &\quad (b^2 - bb_1) + (a^2 - aa_2) \\ = &\quad b^2 + a^2 - 2ab\cos C \end{aligned}$$

So, we can measure the rectangle, summarize our inferences and come up with the law of cosines.

Work your way through this argument, and supply details which you had use to understand it, or had to look up elsewhere.

Problem 6:

Generalize the above argument to work also for an obtuse triangle. Hint: Sometimes you need to add instead of subtract and vice-versa.

[9] This point is called the *orthocenter* of the triangle.