CONTINUED FRACTIONS

PAUL HEWITT, UNIV OF TOLEDO

1. Anthyphairesis

Consider a square of side s and diagonal d. The pythagorean theorem tells us that $d^2 = 2s^2$, or equivalently that $(d/s)^2 = 2$. Are d and s *commensurable*? In other words, is there a unit length u such that d and s are whole multiples of u? If say $d = mu$ and $s = nu$ then $d/s = m/n$. In other words, the ratio d/s would equal a ratio of whole numbers.

But this is impossible. How do we know this? We can use an ancient method called *anthyphairesis* — which means roughly "continually subtract the smaller from the larger" — to explain it. In doing so we will draw a strong link between magnitude and number. Although this method was a favorite of the ancient Greeks, most of them largely ignored this link. They did not conceive of a number system which included both number and magnitude.

Let's start with a simple example, using whole numbers. Say $a = 395368$ and $b = 28397$. We "continually subtract" b from a until what remains is too small to subtract from. In other words, we divide a by b, obtaining a quotient 13 and remainder 26207. Now here is the key to the algorithm: we replace a by b and b by the remainder, and repeat.

$$\begin{gathered}
395368 - 13 \times 28397 = 26207 \\
28397 - 1 \times 26207 = 2190 \\
26207 - 11 \times 2190 = 2117 \\
2190 - 1 \times 2117 = 73 \\
2117 - 29 \times 73 = 0
\end{gathered}$$

What does this tell us? The last line tells us that 73 evenly divides 2117. The line above that implies that $2190 = 2117 + 73$ is also evenly divisible by 73. As we work our way up the chain of equations we find that 73 evenly divides the original pair a and b. In fact, this has produced the *greatest common divisor* of a and b. This is the most efficient method known to compute the gcd. Nowadays it is commonly called "Euclid's algorithm", but this is historically inaccurate, since the method was known centuries before Euclid. We will explore just how efficient anthyphairesis is in the exercises.

We can glean a bit more from the calculations above. If we look at the next-to-last line in the computation above we see an expression for the gcd as a linear combination of 2190 and 2117:

$$73 = 1 \times 2190 - 1 \times 2117.$$

Date: 25 March 2009.

When we substitute into this equation the next line up the chain, we express the gcd as a linear combination of 26207 and 2190:

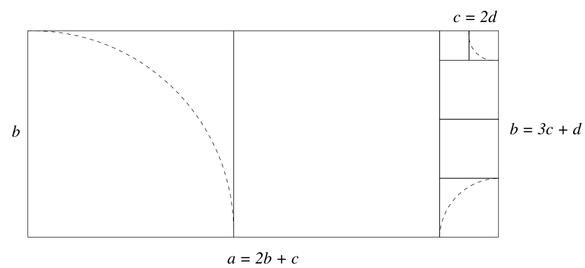
$$73 = 1 \times 2190 - 1 \times (26207 - 11 \times 2190) = 12 \times 2190 - 1 \times 26207.$$

If we continue up the chain we eventually express the gcd as a linear combination of a and b:

$$73 = 12 \times (28397 - 1 \times 26207) - 1 \times 26207 = \cdots = 181 \times 28397 - 13 \times 395368.$$

It turns out that the gcd is the smallest positive linear combination of the given numbers, and is the only one which is also a common divisor.

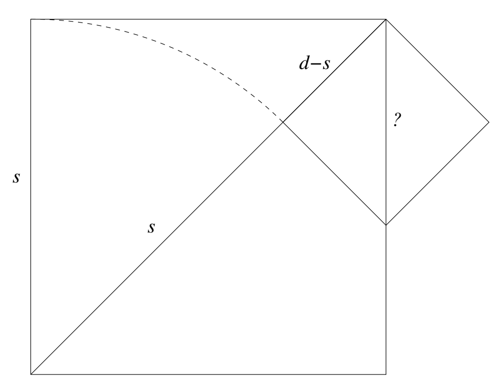
We can visualize anthyphairesis by imagining a and b as lengths in a geometric figure, and then "laying off" one against the other. For example, suppose a and b are the sides of a rectangle:



In the example above, we swing side b around to side a, and find that we can lay it off twice, leaving a length $c = a - 2b$. Now we swing c around to side b, and find that we can lay it of 3 times. This leaves us a length $d = b - 3c$. Finally, we lay off d along c twice exactly, leaving no remainder. The Greeks would say that d *measures* c. That is, if we take d as our unit length, c is exactly 2 units. As with the numerical example above, we can walk the chain backwards: b is $3 \times 22 + 1 = 7$ units, and a is $2 \times 7 + 2 = 16$ units. So, a and b are commensurable — "co-measurable" — with common unit d.

Centuries before Euclid it had been discovered that sometimes this process would never end, and therefore that some pairs of lengths are incommensurable. It is possible that this was discovered by the Pythagoreans, possibly using anthyphairesis and possibly not. We do not know with certainty. Let's return to the case of a square of side s and diagonal d, to see how this *might* have been discovered.

If we lay off s (once) along d we can use the remainder to form a smaller square of side $d - s$:



What is the diagonal of the smaller square? We can figure this out by returning to the pythagorean relation between d and s, namely:

$$\begin{aligned} d^2 &= 2s^2 \\ d^2 - s^2 &= s^2 \\ (d+s)(d-s) &= s^2 \\ \frac{d+s}{s} &= \frac{s}{d-s} \end{aligned}$$

The Greeks would not have arrived at this conclusion using algebra, but rather through geometric reasoning. They would phrase this last relation as follows:

Diagonal-plus-side is to side as side is to diagonal-minus-side.

In symbols, this is expressed nowadays as

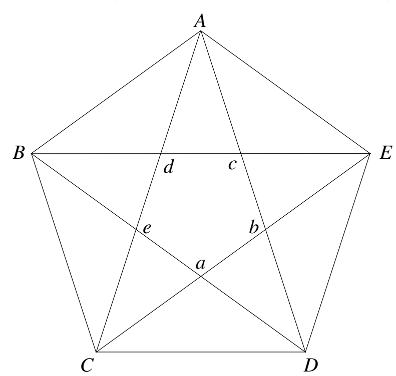
$$d + s : s :: s : d - s.$$

Since all squares are proportional, the ratio we compute in the smaller square equals that computed in the larger square:

$$\frac{? + d - s}{d - s} = \frac{s}{d - s}.$$

Hence the diagonal of the smaller square must be $2s - d$. In particular, any unit which measures both side and diagonal of the larger square must also measure side and diagonal of the smaller square. But we could repeat this process forever, creating smaller and smaller squares all measured by the same unit. This is absurd, and so we must conclude that the side and diagonal of a square are incommensurable.

There is nothing special about a rectangle. To study the relationship of two particular lengths, other geometric figures might be more revealing. Here is a *pentagram*, which had mystical significance to the Pythagoreans:



In the exercises we will explore the ratio of diagonal to side, and in particular compute its anthyphairetic ratio.

There is an ancient legend that the Pythagoreans were blissfully doing mathematics, content that "all is number", and piously avoiding beans. One day someone in the brotherhood (actually, the brother-and-sisterhood: Pythagoras was well ahead of his time in being willing to teach mathematics to women) discovered that the side and diagonal of a square (or possibly the sacred pentagram) are incommensurable. Chaos erupted. Something had to be done. The poor sap was put in a small boat, left to the die in the open sea, his bones picked clean by gulls.

Such legends seemed to be popular a century later in Athens. The Pythagorean society had been overthrown (by locals who wanted to eat beans?) and Greek mathematics had moved away from the naive view that everything can be described in terms of number. The influential teacher Plato had elevated geometry to the pinnacle of mathematics, and insisted that the notions of number and magnitude be kept in separate parts of the brain. Stillwell notes in chapter 1 of *Yearning for the Impossible* that this hampered the natural development of Greek mathematics in several ways:

- It blocked the general idea of algebra.
- It caused mischief with the notion of equality.
- It played havoc with the concept of volume.

All of this may be true, but in the last 30 years or so there has been a re-examination of original sources which has yielded a different consensus about mathematical events of that epoch. Historians note that the Greeks, like the Egyptians who taught them, did not use common fractions, only unit fractions. The process of anthyphairesis allows us to express (or approximate) ratios using only unit fractions, without the need for a theory of common fractions.

For example, consider the ratio "diagonal-plus-side to side" above. We learned that s can be laid off along d once, and hence along $d + s$ twice. The remainder is $d - s$, which is itself the side of a smaller square, whose "diagonal-plus-side" is s. As with our numerical example, we can replace the $d + s$ and s with s with $d - s$, respectively, to get an infinitely continued fraction:

$$\begin{aligned}\frac{\text{diagonal-plus-side}}{\text{side}} = \frac{d+s}{s} &= 2 + \frac{d-s}{s} \\ &= 2 + \cfrac{1}{\cfrac{\text{diagonal-plus-side}}{\text{side}}} \\ &= 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}\end{aligned}$$

Continued fractions are very cool. We will explore one of their uses in the next section. Every real number has an expansion as a continued fraction. For example,

$$\pi = \frac{\text{circumference}}{\text{diameter}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\ddots}}}}}$$

In other words $\pi \approx 3\frac{1}{7}$. But that is a bit too big: so replace 7 by $7\frac{1}{15}$ to be more precise. But now that is too small: so replace 15 by $15\frac{1}{1}\ldots$ If we stop there we get the approximation

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113}.$$

This is the amazingly accurate approximation first computed by Tsu Ch'ung-Chih in the fifth century. It is not known what method he used. It is not a coincidence that the continued fraction produced such accuracy. It is known that continued fractions always produce the best rational approximation for a given size of denominator. Continued fractions are more accurate than decimal approximations!

If it's true that continued fractions are better approximations than decimals, why do we still use decimals to represent real numbers? Until fairly recently it was not known how to perform ordinary arithmetic operations using this representation, without first converting to decimals. We now know how to do this: see `http://www.inwap.com/pdp10/hbaker/hakmem/cf.html` for details. These new methods are probably not yet ready to be taught in elementary schools, but they do offer some possible improvements in computer architecture — which is where most arithmetic is performed these days.

Altho manipulating continued fractions is a bit complicated, generating them is dead easy. There are only two steps, repeated until you have as much accuracy as you need. Start with any real number x.

- **Split** x into its integer and fractional parts.
- **Invert** the fractional part.

The **invert** step has produced a new real number, and you simply repeat these two steps to it. If the fractional remainder is ever 0, stop — x is rational, and the continued fraction is finite. In any case, the **split** steps have generated a (possibly infinite) sequence of integers $a_0, a_1, a_2, \ldots$, with the property that

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

Here is the first few steps of a simple example:

$$\begin{aligned}
x = 2.718 &= 2 + \frac{718}{1000} && \text{(split)}\\
&= 2 + \cfrac{1}{\cfrac{1000}{718}} && \text{(invert)}\\
&= 2 + \cfrac{1}{1 + \cfrac{282}{718}} && \text{(split)}\\
&= 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{718}{282}}} && \text{(invert)}\\
&= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{154}{282}}} && \text{(split)}
\end{aligned}$$

Already the usual notation for fractions is making it cumbersome to write out the steps. This obscures the essential simplicity. Many authors have introduced simpler notation, but there is no standard notation. Here is how we will write things: "+/" will mean "plus the reciprocal of", where this phrase refers to to everything to the right of the "+/" symbol. Here is what the calculation above looks like in the new, simpler notation:

$$
\begin{aligned}
x = 2.718 = 2 + \frac{718}{1000} & \qquad \text{(split)} \\
= 2 + /\frac{1000}{718} & \qquad \text{(invert)} \\
= 2 + /1 + \frac{282}{718} & \qquad \text{(split)} \\
= 2 + /1 + /\frac{718}{282} & \qquad \text{(invert)} \\
= 2 + /1 + /2 + \frac{154}{282} & \qquad \text{(split)}
\end{aligned}
$$

If you look closely you see that these are the same steps we would use to compute gcd(2718, 1000): division with remainder, followed by transposing arguments. Keep this in mind as we finish this example:

$$
\begin{aligned}
x = 2 + /1 + /2 + /\frac{282}{154} & \qquad \text{(invert)} \\
= 2 + /1 + /2 + /1 + \frac{128}{154} & \qquad \text{(split)} \\
= 2 + /1 + /2 + /1 + /\frac{154}{128} & \qquad \text{(invert)} \\
= 2 + /1 + /2 + /1 + /1 + \frac{26}{128} & \qquad \text{(split)} \\
= 2 + /1 + /2 + /1 + /1 + /\frac{128}{26} & \qquad \text{(invert)} \\
= 2 + /1 + /2 + /1 + /1 + /4 + \frac{24}{26} & \qquad \text{(split)} \\
= 2 + /1 + /2 + /1 + /1 + /4 + /\frac{26}{24} & \qquad \text{(invert)} \\
= 2 + /1 + /2 + /1 + /1 + /4 + /1 + \frac{2}{24} & \qquad \text{(split)} \\
= 2 + /1 + /2 + /1 + /1 + /4 + /1 + /\frac{24}{2} & \qquad \text{(invert)} \\
= 2 + /1 + /2 + /1 + /1 + /4 + /1 + /12 &
\end{aligned}
$$

Since we started with a rational number, namely 2718/1000, the process eventually comes to a halt. The resulting expression involves only unit fractions. What would it look like if we were to recombine the pieces into a single common fraction?

Would we end up with the original fraction? Let's take a look.

$$\begin{aligned}
x &= 2+/1+/2+/1+/1+/4+/1+/12\\
&= 2+/1+/2+/1+/1+/4+/\frac{13}{12}\\
&= 2+/1+/2+/1+/1+/\frac{4\cdot 13+12}{13}\\
&= 2+/1+/2+/1+/1+/\frac{64}{13}\\
&= 2+/1+/2+/1+/\frac{77}{64}\\
&= 2+/1+/2+/\frac{141}{77}\\
&= 2+/1+/\frac{2\cdot 141+77}{141}\\
&= 2+/1+/\frac{359}{141}\\
&= 2+/\frac{500}{359}\\
&= \frac{1359}{500}
\end{aligned}$$

This is not so surprising: it is the original fraction reduced to lowest terms.

But something very interesting happens when you apply the same process to an irrational number. For example, $2.718 \approx e$. What if we compute the continued fraction expansion of e? Would it be essentially the same? If we use a calculator we find that

$$e = 2+/1+/2+/1+/1+/4+/1+/1+/\cdots$$

The first difference is in the eighth term. How does this affect the resulting common fraction?

$$e \approx 2+/1+/2+/1+/1+/4+/1+/1 = \frac{193}{71}.$$

This is an approximation with a much smaller denominator than 500. How good is it? It is accurate to three decimal places. In fact the error is about 2.8×10^{-5}, which roughly 10 times smaller than the error in the approximation $e \approx 2.718$!

There are many practical applications of this sort of accuracy. Altho the calculations are tedious for humans, the end result has big payoffs. For computers, these calculations are just as routine as those involving decimals.

2. Brahmagupta's equation

Many centuries after Euclid and many thousands of miles away, the great Indian mathematician Brahmagupta was challenging his students with problems like this:

> Ninety-two squares augmented by one is a square.

Even though India gave birth to our decimal notation, Indian mathematicians expressed their mathematics rhetorically, including numbers. In fact, their textbooks were poetic — they rhymed! In modern notation, he was asking for whole-number solutions to the indeterminate quadratic

$$92x^2 + 1 = y^2. \tag{1}$$

Brahmagupta did not find a general solution to problems like this, but he did discover the important underlying structure. We now call this a "multiplicative group". Brahmagupta called it "compounding".

What was Brahmagupta's rule for compounding?

> If x_1 and y_1 give a solution for additive c_1 and x_2 and y_2 give a solution for additive c_2 then $y_1x_2 + y_2x_1$ and $y_1y_2 + 92x_1x_2$ give a solution for additive c_1c_2.

What does this mean? What is an "additive" parameter?

The idea is to locate values of x and y that solve an equation similar to (1), but possibly with a different additive constant, other than 1. You can then compound them using the formulas above in the hopes of getting an equation in which the terms have a common factor. After you factor this out you might be closer to additive 1, your ultimate goal.

For example, the squares nearest 92 are 81 and 100. So $1, 9$ and $1, 10$ give solutions with additive -11 and additive 8, respectively. That is,

$$92 \cdot 1^2 - 11 = 9^2$$
$$92 \cdot 1^2 + 8 = 10^2$$

Brahmagupta's compounding rule tells us that

$$92 \cdot (9 \cdot 1 + 10 \cdot 1)^2 - 88 = (9 \cdot 10 + 92 \cdot 1 \cdot 1)^2.$$

Why does Brahmagupta's rule work? To get the idea behind compounding, it is best to rewrite the problem:

$$y^2 - 92x^2 = 1. \tag{2}$$

The quantity on the left is called the *norm.* We can factor the norm as a difference of squares:

$$N(y + \sqrt{92}x) = y^2 - 92x^2 = (y + \sqrt{92}x)(y - \sqrt{92}x). \tag{3}$$

The two factors are said to be *conjugates* of one another. Conjugation is often denoted with an overline:

$$\overline{y + \sqrt{92}x} = y - \sqrt{92}x. \tag{4}$$

The first key property is that expressions of this type are closed under the arithmetic operations of addition and multiplication. For example,

$$(y_1 + \sqrt{92}x_1)(y_2 + \sqrt{92}x_2) = (y_1y_2 + 92x_1x_2) + \sqrt{92}(y_1x_2 + y_2x_1).$$

The formulas for the two components of the product are precisely Brahmagupta's rules for compounding.

The second key property is that the conjugate and the norm are *homomorphisms* of the multiplicative structure. This means that if u and v are expressions of the form $x + \sqrt{92}y$, where x and y are rational, then

$$\overline{uv} = \overline{u} \cdot \overline{v}. \tag{5}$$

$$N(uv) = N(u)N(v). \tag{6}$$

You are asked to verify these identities in the exercises.

Let's walk thru Brahmagupta's solution of (1), or equivalently of (2). If we take the solution $1, 10$ for additive 8 and compound it with itself, we get the solution $20, 192$ for additive 64. In symbols:

$$N(10 + \sqrt{92}) = 8 \implies N((10 + \sqrt{92})^2) = N(192 + 20\sqrt{92}) = 64.$$

If we divide this solution by 4 — which also divides the conjugate by 4 — then we get the solution $5, 48$ for additive 4:

$$N((192 + 20\sqrt{92})/4) = N(48 + 5\sqrt{92}) = 64/4^2 = 4.$$

If we compound this solution with itself we get the solution $480, 4604$ for additive 16:

$$N((48 + 5\sqrt{92})^2) = N(4604 + 480\sqrt{92}) = 4^2 = 16.$$

Finally, divide this solution by 4 to obtain the solution $120, 1151$ for additive 1:

$$N((4604 + 480\sqrt{92})/4) = N(1151 + 120\sqrt{92}) = 16/4^2 = 1.$$

Notice something else that compounding makes clear: once we obtain a solution for additive 1 we get infinitely many of them, since we can repeatedly compound the solution with itself as many times as we like.

This was a clever solution of this particular problem, but unfortunately it does not always work. The general case was first solved completely by Acarya Jayadeva, sometime in the late 10th or early 11th century, but was not well understood.

About a century or so after Acarya Jayadeva there lived probably the greatest medieval Indian mathematician, Bhaskara. Actually there were two Bhaskaras in Indian mathematical history. The first was a contemporary of Brahmagupta. The second and more accomplished is often known as Bhaskaracharya, or "Bhaskara the Teacher". Bhaskara's solution of Brahmagupta's equation was much clearer and more understandable than that of Acarya Jayadeva.

Brahmagupta's equation was first studied in Europe by the great Fermat. There is as yet no evidence that he knew of any of the work of Indian mathematicians, at least not directly. However Fermat did study from translations of Arabic texts, some of which might have contained some of the work of Brahmagupta or Bhaskara. On the other hand, most of the Arabic works that found their way into European translations came from Spain, which was not in touch with the latest developments from Baghdad and further East.

At any rate, Fermat challenged his peers in Britain, France, Germany, Italy, and the Netherlands to find a general whole-number solution of the equation

$$y^2 = Dx^2 \pm 1, \tag{7}$$

where D is a given integer whose root $\sqrt{D}$ is irrational. One of Fermat's correspondents was the Englishman John Pell. Euler mistakenly thought Pell had solved (all or part of) the problem, and so the equations (7) have been called Pell's equations ever since, despite the priority of Brahmagupta, Acarya Jayadeva, and Bhaskara.

The first complete solution *with proof* of its validity was given by Lagrange in the 1760s. The first proof that Bhaskara's method also always gives a solution did not appear until 1929.

We will not describe Bhaskara's solution. It is described in section 6.7 of Katz' *A History of Mathematics* (2nd edition). Instead we will describe Lagrange's solution, which uses continued fractions. We will not, however, attempt to provide even a sketch of the proof that it works.

We will illustrate Lagrange's method with one of Bhaskara's examples, from his famous book *Lilavati*:

$$67x^2 + 1 = y^2. \tag{8}$$

To see why we might not be too surprised to see continued fractions come into the picture, let's rewrite equation (8):

$$67 + \frac{1}{x^2} = \frac{y^2}{x^2}.$$

Thus, any solution of Bhaskara's equation also gives a rational approximation of $\sqrt{67}$, with an error that decreases with the size of the denominator x. We know that the best rational approximations of this type are given by continued fractions. Hence it is natural to compute the continued fraction expansion of $\sqrt{67}$.

But the continued fraction of a quadratic irrational is infinite periodic. Where do we stop computing? As soon as it starts to repeat. This will give a solution with additive ± 1, depending on whether or not the period is odd or even. If it is odd we can simply compound the solution with itself to obtain a solution with additive 1.

OK, that's the strategy. Let's get to work ... but first we make one technical adjustment which makes it easier to recognize when the continued fractions starts to repeat. It turns out to be a bit easier to compute the continued fraction of $\sqrt{67}+8$ — that is, $\sqrt{67}$ plus its integer part. Now finally we are ready to compute. We use the notation introduced in the examples above.

$$
\begin{aligned}
\sqrt{67}+8 &= 16+(\sqrt{67}-8) && \text{(split)}\\
&= 16+/\tfrac{1}{3}(\sqrt{67}+8) && \text{(invert)}\\
&= 16+/5+\tfrac{1}{3}(\sqrt{67}-7) && \text{(split)}\\
&= 16+/5+/\tfrac{1}{6}(\sqrt{67}+7) && \text{(invert)}\\
&= 16+/5+/2+\tfrac{1}{6}(\sqrt{67}-5) && \text{(split)}\\
&= 16+/5+/2+/\tfrac{1}{7}(\sqrt{67}+5) && \text{(invert)}\\
&= 16+/5+/2+/1+\tfrac{1}{7}(\sqrt{67}-2) && \text{(split)}\\
&= 16+/5+/2+/1+/\tfrac{1}{9}(\sqrt{67}+2) && \text{(invert)}\\
&= 16+/5+/2+/1+/1+\tfrac{1}{9}(\sqrt{67}-7) && \text{(split)}\\
&= 16+/5+/2+/1+/1+/\tfrac{1}{2}(\sqrt{67}+7) && \text{(invert)}\\
&= 16+/5+/2+/1+/1+/7+\tfrac{1}{2}(\sqrt{67}-7) && \text{(split)}\\
&= 16+/5+/2+/1+/1+/7+/\tfrac{1}{9}(\sqrt{67}+7) && \text{(invert)}\\
&= 16+/5+/2+/1+/1+/7+/1+\tfrac{1}{9}(\sqrt{67}-2) && \text{(split)}\\
&= 16+/5+/2+/1+/1+/7+/1+/\tfrac{1}{7}(\sqrt{67}+2) && \text{(invert)}\\
&= 16+/5+/2+/1+/1+/7+/1+/1+\tfrac{1}{7}(\sqrt{67}-5) && \text{(split)}\\
&= 16+/5+/2+/1+/1+/7+/1+/1+/\tfrac{1}{6}(\sqrt{67}+5) && \text{(invert)}\\
&= 16+/5+/2+/1+/1+/7+/1+/1+/2+\tfrac{1}{6}(\sqrt{67}-7) && \text{(split)}\\
&= 16+/5+/2+/1+/1+/7+/1+/1+/2+/\tfrac{1}{3}(\sqrt{67}+7) && \text{(invert)}\\
&= 16+/5+/2+/1+/1+/7+/1+/1+/2+/5+\tfrac{1}{3}(\sqrt{67}-8) && \text{(split)}\\
&= 16+/5+/2+/1+/1+/7+/1+/1+/2+/5+/(\sqrt{67}+8) && \text{(invert)}
\end{aligned}
$$

The period here is 10 — there were 10 split-invert steps before we returned to the original expression — and so we should obtain a solution for additive 1.

Now that we have the continued fraction, we truncate it after one period and rewrite the expression as a single common fraction:

$$\begin{aligned}
\sqrt{67}+8 &\approx 16+/5+/2+/1+/1+/7+/1+/1+/2+/5 \\
&= 16+/5+/2+/1+/1+/7+/1+/1+/\frac{11}{5} \\
&= 16+/5+/2+/1+/1+/7+/1+/\frac{16}{11} \\
&= 16+/5+/2+/1+/1+/7+/\frac{27}{16} \\
&= 16+/5+/2+/1+/1+/\frac{205}{27} \\
&= 16+/5+/2+/1+/\frac{232}{205} \\
&= 16+/5+/2+/\frac{437}{232} \\
&= 16+/5+/\frac{1106}{437} \\
&= 16+/\frac{5967}{1106} \\
&= \frac{96578}{5967}
\end{aligned}$$

Next, we subtract 8 to get the approximation of $\sqrt{67}$.

$$\sqrt{67} \approx \frac{96578}{5967} - 8 = \frac{48842}{5967}.$$

The numerator and denominator should give a solution of Bhaskara's problem (8):

$$N(48842 + 5967\sqrt{67}) = 48842^2 - 5967^2 \cdot 67 = 1.$$

Magically, they do!

Exercises

(1) Use anthyphairesis to show that $\gcd(37, 11) = 1$. Use graph paper to illustrate this, starting with a 37×11 rectangle. Finally, use the results of your calculation to find a linear combination of 37 and 11 which equals 1.

(2) Use anthyphairesis to compute $\gcd(70016481639, 4058897373)$ and to express the gcd as a linear combination of the parameters.

(3) Consider two steps of anthyphairesis:

$$a - q_1 \times b = r_1$$
$$b - q_2 \times r_1 = r_2$$

Show that there are exactly two possibilities:

i. either $q_1 = 1$, $b > \frac{1}{2}a$, and $r_1 < \frac{1}{2}a$;

ii. or else $q_1 > 1$, and $r_1 < b \leq \frac{1}{2}a$.

Hence in any case, after two steps we obtain a pair r_1, r_2 with fewer binary bits than a, b. For example, if a and b each have 1000 binary bits then anthyphairesis requires at most 2000 steps.

(4) The "worst case" scenario for anthyphairesis is when the quotient is 1 at each step. In this case, anthyphairesis will require a lot of steps. Each of the parts below explores this situation, and each is worth 4 points.

(a) Generate 10 terms $F_0 = 0, F_1 = 1, \ldots, F_9$ of the Fibonacci sequence. Apply anthyphairesis to the last two terms. What do you find? Is this a general pattern? Explain!

(b) Prove by induction that if a and b are whole numbers and anthyphairesis requires more than n steps to arrive at $\gcd(a, b)$ then $\min(a, b) > F_{n+1}$.

(5) Forget about anthyphairesis for a moment, and simply define x to be

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

Show that $x = 2 + 1/x$, and hence that $x = 1 + \sqrt{2}$. Determine y and z, where we define

$$y = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

and

$$z = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

(6) Use the symmetry of the pentagram to prove the following.
   (a) $\angle ABe = \angle AeB$.
   (b) $\triangle ABe$ is isosceles.
   (c) $\triangle ABd$ is similar to $\triangle ecd$.
   (d) $\overline{eC} : \overline{AB} :: \overline{ac} : \overline{ab}$.

(7) Use the previous problem to compute that

$$\overline{AC} : \overline{AB} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

Are the diagonal and side of a regular pentagon commensurable? Justify your answer.

(8) Compute the continued fraction expansion of 104348/33215 (by hand) and the first 9 terms of the continued fraction expansions of π and e (using a calculator or Maple). Be careful! Roundoff errors can be a problem. Check your answer, by searching online. If your calculator is not accurate enough for the task, use Maple or some other device which allows you to adjust the accuracy as needed.

(9) Investigate the life and mathematics of Eudoxus. Explain his theory of proportion. Specifically, how did he define the notion that "four magnitudes are in the same ratio, the first to the second and the third to the fourth" without defining what he meant by ratio? Investigate the life and mathematics of Richard Dedekind. In what way did he claim that his work on "cuts" of the real line was based on the work of Eudoxus?

(10) In this problem you will prove that conjugation and norm are multiplicative.
   (a) Use definition (4) to prove identity (5).
   (b) Use definition (3) and part (a) to prove identity (6).

(11) Use Lagrange's method to find solutions of the equations $y^2 - Dx^2 = 1$, for each of the following D. Once you have one solution for each, compound the solution with itself to obtain a second solution. Verify that your solutions satisfy Pell's equation.
   (a) $D = 23$.
   (b) $D = 41$.
   (c) $D = 92$.

(12) Read about Bhaskara's method, called *chakravala*, in Katz' book cited above. Use it to solve the first two equations in the previous problem. (This problem is worth double points.)

UNIV OF TOLEDO