Class Notes on Toeplitz Sequences

Heidi Tucker, Matt Marks, and Christina Lucuta, 2006, revised 28apr11 by GF

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1 Toeplitz Sequence

A Toeplitz Sequence begins by giving an increasing sequence of rational numbers $s_1 \leq s_2 \leq s_3 \leq ....$ This matches up with another sequence of rationals t_n that moves in a descending order: $... \leq t_3 \leq t_2 \leq t_1$. To this we add a requirement that

$$(\forall i, j)(s_i \leq t_j)$$

. But a gap is unwanted and can be avoided by "squeezing the numbers". We add a final requirement that

$$\lim_{x \to \infty} t_n - s_n = 0$$

.

Basically, every real number is defined by a Toeplitz sequence, namely its decimal expansion. For example, $\sqrt{2} = 1.41459....$ defines a T-sequence as follows.

With this decimal one can take

$$1 < 1.4 < 1.41 < 1.414 < .... < 1.415 < 1.42 < 1.5 < 2$$

And every time it is the next digit in the sequence that is considered. From the above statement, it is known that $t_1 - s_1 \leq 1$ and that $t_2 - s_2 \leq \frac{1}{10}$. From this it is possible to conclude that $t_n - s_n \leq \frac{1}{10^{n-1}}$. It is also easy to see here that every $s_i \leq t_j$.

2 The Story of e

From there the class went to the story of e and compound interest. In order to compute interest there must be an interest rate x, and a principal a_0. Assuming somebody's interest is compounded yearly, the principal will advance to $a_1 = a_0 + xa_0 = a_0(1+x)$ at the end of the year. The next time around it would look like $a_2 = a_1 + xa_1 = a_1(1+x) = a_0(1+x)^2$. Interest can be compounded quarterly as well. To compound quarterly throughout the year take

$$a_1 = a_0(1+\frac{x}{4}), a_2 = a_0(1+\frac{x}{4})^2, a_4 = a_0(1+\frac{x}{4})^4.$$

a_4 represents the end of the year. In order to compound monthly take:

$$a_1 = a_0(1+\frac{x}{12})$$

And at the end of the 12 month cycle the final product will look like

$$a_{12} = a_0(1+\frac{x}{4})^{12}$$

The more something is compounded the larger the return on interest is. If a person had a contract where interest were to be compounded every day of the year, he or she would earn more money than a person who has had his or her interest compounded every month. But by looking at the three above examples of compounding interest it is easy to see that they all have something in common. That is the expression of the form

$$(1+\frac{x}{n})^n.$$

2.1 The special case that $x = 1$.

Simplifiying for now with $x = 1$ we next demonstrate the existence of Toeplitz sequence

$$s_1 \le s_2 \le s_3 \le ... \le t_3 \le t_2 \le t_1$$

where we define

$$s_n = (1+\frac{1}{n})^n = (\frac{n+1}{n})^n$$

and choose a suitable upper sequence t_n for which we will show that $t^{n+1} - s_n$ is a null sequence.

The completeness axiom of the reals as expressed in terms of Toeplitz Sequences then permits us to conclude that $\lim_{n\to\infty}(1+\frac{x}{n})^n$ exists. We give this famous real number the name e after Euler.
So if $s_n = (1+\frac{1}{n})^n$, then $s_1 = (1+\frac{1}{1})^1 = 2$ and $s_2 = (1+\frac{1}{2})^2 = \frac{9}{4} = 2.25$ and $s_3 = (1+\frac{1}{3})^3 = 2.37$.
Lemma: For all $n \in \mathbb{N}$,

$$(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}.$$

This is worked out by realizing that $\frac{1}{n} > \frac{1}{n+1}$. Then when 1 is added to each side the sum is $(1+\frac{1}{n}) > (1+\frac{1}{n+1})$. But once these are taken to their respected powers there comes a problem.

$$s_n = (1+\frac{1}{n})^n > (1+\frac{1}{n+1})^n < (1+\frac{1}{n+1})^{n+1} = s_{n+1}.$$

Because of these opposing inequalities, we cannot conlude that the s_n are monotonically increasing, at least not yet.
The next thing to do is work with t_n.

$$t_n = (1-\frac{1}{n})^{-n} = \frac{1}{(1-\frac{1}{n})^n}$$

If $(1+\frac{1}{n}) = \frac{n+1}{n}$ is taken, the reciprocal of $\frac{n+1}{n}$ is taken to be $(\frac{n}{n+1})^{-1}$ and the numerator adds and subtracts a 1 to cancel things out. So,

$$(\frac{(n+1)-1}{n+1})^{-1}$$

then becomes

$$(1-\frac{1}{n+1})^{-1}.$$

So,

$$(1+\frac{1}{n})^{n+1} = (1-\frac{1}{n+1})^{-(n+1)} = t_{n+1}.$$

Taking $s_n + \frac{1}{n}s_n = t_{n+1}$ implies $s_n < t_{n+1}$.
Once we have actually proved the monotonicity of both sequences, we can to find the separation of $s_i \leq t_j$. We need the monoticity of the t_n to find an upper bound here

$$t_{n+1} - s_n = \frac{1}{n}s_n \leq \frac{1}{n}t_1 \to 0$$

from which we conclude that $\lim_{n\to\infty}(t_{n+1}-s_n) = 0$. (Recall that the product of null and a bounded sequence is null.)

2.2 A Geometric Inequality

2.3 Motivation

There is no good motivation. It is a truly ingenious application of one branch of mathematics to another. This is the nature of mathematics and this course is supposed to give you little glimpse of this every so often.

2.4 Factoring the difference of powers.

Now with a little digression into some geometric sums, recall how to prove that that $x+x^2+...+x^n = \frac{x}{1-x}$ by multiplying both sides by $(1-x)$. Replace $1, x$ by a, b and multiplying

$$a^n + a^{n-1}b + a^{n-2}b^2 + a^{n-3}b^3 + ... + a^2b^{n-2} + ab^{n-1} + b^n$$

by $(a-b)$, yields

$$a^{n+1} + a^n b + a^{n-1}b^2 + ... - a^n b - a^{n-1}b^2 - ... - b^{n+1} = a^{n+1} - b^{n+1}.$$

Therefore,

$$\frac{a^{n+1} - b^{n+1}}{a-b} = a^n + a^{n-1}b + a^{n-2}b^2 + ... + ab^{n-1} + b^n$$

All we want from this identity is the two inequalities for $a < b$ that

$$(n+1)a^n < \frac{a^{n+1} - b^{n+1}}{a-b} < (n+1)b^n$$

We take s_n and t_n and rewrite them simplify doing arithmetic with them:

$$s_n = (1 + \frac{1}{n})^n = (\frac{n+1}{n})^n$$

$$t_n = (1 - \frac{1}{n})^{-n} = (\frac{n-1}{n})^{-n} = (\frac{n}{n-1})^n$$

Some examples for s_n:

$$s_2 = (\frac{3}{2})^2$$

$$s_3 = (\frac{4}{3})^3$$

$$s_4 = (\frac{5}{4})^4$$

Some examples for t_n:

$$t_3 = (\frac{3}{2})^3$$

$$t_4 = (\frac{4}{3})^4$$

$$t_5 = (\frac{5}{4})^5$$

2.5 Apply the Geometric Inequality

Look at the Geometric Inequality above, with $b = \frac{n+1}{n} > a = \frac{n+2}{n+1}$.

$$\frac{(1+\frac{1}{n})^{n+1} - (1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n}) - (1+\frac{1}{n+1})} \leq (n+1)(1+\frac{1}{n})^n$$

Transform this lemma so that s_n is integrated into it using $(n+1)(1+\frac{1}{n})^n = (n+1)s_n$ by following the steps:

(1) $\frac{s_n(1+\frac{1}{n})-s_{n+1}}{\frac{1}{n}-\frac{1}{n+1}}$

(2) $\frac{s_n(1+\frac{1}{n})-s_{n+1}}{\frac{n+1-n}{n(n+1)}}$

(3) $\frac{s_n(1+\frac{1}{n})-s_{n+1}}{\frac{1}{n(n+1)}}$

(4) $(s_n(1+\frac{1}{n}) - s_{n+1})(n(n+1))$

(5) $s_n(1+\frac{1}{n}) - s_{n+1} \leq \frac{n+1}{n(n+1)}s_n$

Watch closely, the underline portions will cancel out...

(6) $s_n + \underline{\frac{1}{n}s_n} - s_{n+1} \leq \underline{\frac{1}{n}s_n}$

(7) $s_n \leq s_{n+1}$

2.6 In summary

We have applied the Geometric Inequality to find that

$$\frac{1}{n}(\frac{n+2}{n+1})^n < (\frac{n+1}{n})^{n+1} - (\frac{n+2}{n+})^{n+1} < \frac{1}{n}(\frac{n+1}{n})^n$$

Rewrite the Middle and RHS in terms of the s sequence we have

$$s_n(1+\frac{1}{n}) - s_{n+1} < s_n\frac{1}{n} \Rightarrow s_n < s_{n+1}.$$

The Middle can also be rewritten in terms of the t sequence thus

$$0 < t_{n+1} - t_{n+2}(1+\frac{1}{n+1} \Rightarrow t_{n+1} < t_{n+2}.$$

Note that we did use that the LHS, as well as t_{n+2} is positive.

2.7 Continuing with the original exposition.

Using this reasoning, how can we determine that $t_{n+1} \leq t_n$? WELL.... During class we first determined what a and b were.

$$a^{n+1} = t_n(1-\frac{1}{n})^{-1} = (1-\frac{1}{n})^{-1}(1-\frac{1}{n})^{-1} = (\frac{n}{n-1})^n(\frac{n}{n-1}) = (\frac{n}{n-1})^{n+1}$$

Thus, because $a^{n+1} = (\frac{n}{n-1})^{n+1}$, $a = \frac{n}{n-1}$. To find b:

$$b^{n+1} = t_{n+1} = (1+\frac{1}{n+1})^{-(n+1)} = (\frac{n+1}{n})^{n+1}$$

Thus, because $b^{n+1} = (\frac{n+1}{n})^{n+1}$, $b = \frac{n+1}{n}$.

Simply as a checker, we can set these in a relation and cross multiply by positive numbers so as not to change the relation. We find which of the two, $a = \frac{n}{n-1}$ or $b = \frac{n+1}{n}$ is larger.

$$\frac{n}{n-1} \sim \frac{n+1}{n}$$

Which gives us

$$n^2 \sim n^2 - 1.$$

This, quite obviously shows that a is larger than b. Using the earlier lemma again,

$$(n+1)a^n \leq \frac{(1+\frac{1}{n})^{n+1} - (1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n}) - (1+\frac{1}{n+1})} \leq (n+1)(1+\frac{1}{n})^n$$

While using

$$t_n = (1-\frac{1}{n})^{-n} = (\frac{n-1}{n})^{-n} = (\frac{n}{n-1})^n$$

We can craftily incorporate t_n in like such:

$$n(\frac{n+1}{n})(\frac{n+1}{n})^n \leq \frac{t_n(1-\frac{1}{n})^{-1} - t_{n+1}}{(\frac{n}{n-1}) - (\frac{n+1}{n})} \leq (n+1)(\frac{n}{n-1})^n$$

We can eventually determine $t_{n+1} \leq t_n$.
Exercise: Write the conclusion, pulling all the parts of this proof together.

3 Limits of increasing sequences.

Let's start the discussion by looking at

$$(1+\frac{1}{n})^k \rightarrow ? \text{ as } n \rightarrow \infty$$

Exercise: Find the limit of $(1+\frac{1}{n})^k$, k $\in \mathbb{R}$
Continue with the original problem, where $k \in \mathbb{N}$.

$$1+\frac{1}{n} \rightarrow 1$$

because

$$\frac{1}{n} \rightarrow 0$$

then

$$(1+\frac{1}{n})^2 \rightarrow 1^2$$

because the product of limits is the limit of products, which itself follows from the limiting theorem "null * null = null"

By Induction: Assume $(1+\frac{1}{n})^k \to 1$, then show $(1+\frac{1}{n})^{k+1} \to 1$

PROOF:
$(1+\frac{1}{n})^{k+1} = (1+\frac{1}{n})^k(1+\frac{1}{n}) \to 1 \times 1$ (product of lims)

Proposition: Given $0 < a_1 < a_2 < ...$, but it is bounded above

Then $\lim_{n\to\infty} a_n = a \leq k$
Proof: There are two ways a positive sequence can fail to have a limit. It can march to infinity, or it can get closer and closer to more than one number. The limit is defined to be unique. The latter case cannot be the case for an increasing sequence. (Can you supply an argument for this step?) And it cannot march to infinity because we are assuming the sequence is bounded.
Note: This is not an entirely satisfactory proof, is leaves too much un supported. The correct proof is to actually find this limit, and that would be the *least upper bound* of the sequence. But for such a proof, we would first have to study the Least Upper Bound property, which is actually an axiom for the reals. So we have to accept the intiutive proof for now.

Example: An example that is not one of the theorem above

$$a_m = \begin{bmatrix} 1+\frac{1}{m} \text{ when } m \text{ is even} \\ k-\frac{1}{m} \text{ when } m \text{ is odd} \end{bmatrix}$$

If k=1, then a_m converges to 1

If $k\neg = 1$, then this sequence does not converge. But it is also not an increasing sequence, so "increasing" is a sufficient (but not necessary) reason for a bounded sequence to converge.