Seminar on Bounded Sets, beginning SP11-R10

Prepare your solutions to these problems to be done at the board in seminar. Be sure you have readable solutions in your Journal.

1. Which of the following sets have an upper bound, and which have a lower bound? In the cases where these exist, state what the least upper bound bounds and the greatest lower bounds are. Can you prove your assertions?
 a $A = \{-1, 3, 7, -1\}$
 b $B = \{x \in \mathbb{R} || x - 3| < |x + 7|\}$
 c $C = \{x \in \mathbb{R} | x^3 - 3x < 0\}$
 d $D = \{x \in \mathbb{N} | x^2 = a^2 + b^2$ for some $a, b \in \mathbb{N}\}$
2. Write down proofs of the following statements about subsets of $\mathbb{R}$.
 a If x is an upper bound for A, and $x \in A$, then x is the least upper bound for A.
 b $(\forall a \in B)(y \leq a) \wedge (y \in B) \Rightarrow (y = glb(B))$.
 c If $A \subset B$ then a lower bound for B is also a lower bound for A.
 d If $A \subset B$ and $a = glb(A)$ and $b = glb(B)$ then $a \leq b$.
3. Prove a set of real numbers cannot have more than one least upper bound.
4. Find the lub and glb of the following sets:
 a $\{x | x = 2^{-p} + 3^{-q}$ for some $p, q \in \mathbb{N}\}$
 b $\{x \in \mathbb{R} | 3x^2 - 4x < 1\}$
 c the set of all real numbers between 0 and 1 whose decimal expansion contains no nines.
5 Construct examples for the following
 a A set of rationals having rational least upper bound.
 b A set of rationals having irrational least upper bound.
 a A set of irrationals having rational least upper bound.
6 Which of the following statements are true and which are false? Give adequate reasons for your answer.
 a Every set of real numbers has a glb.
 b $(\forall r \in \mathbb{R})(\exists B \subset \mathbb{Q})(r = glb(B))$
 c Let $C, D \subset \mathbb{R}$. Define $CD = \{cd | c \in C$ and $d \in D\}$. Then $c = glb(C)$ and $d = glb(D)$ only if $cd = glb(CD)$.
 d If the greatest lower bound of a set of real numbers exists but is not a member of the set, then the set must be infinite, and have a subsequence that converges to its greatest lower bound.
7. Prove that the cubic equation $x^3 - x - 1 = 0$ has a real root by showing that any root of the equation is the least upper bound of a suitable set.
8. Let $S = \{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $T_n = \{x_m\}_{m=n}^{\infty}$. Assume that T_1 has a lower bound. Deduce that $(\forall n)(\exists b_n)(b_n = glb(T_n)) \Rightarrow b_1 \leq b_2 \leq b_3 \ldots.$

8 continued For the following sequences find the b_n and find the lub of the b_n if it exists.
 a $x_n = n$
 b $x_n = \frac{1}{n}$
 c $x_n = 1 + (-1)^n$