EXERCISES

Words like "determine", "show", "obtain", or "construct" include a request r justification; these are very similar to "prove". Answers to problems in this ook should be given full explanations. Explanations include *sentences*; reasoning nnot be explained without words.

Easier problems are indicated by "(–)", harder problems by "(+)". Those signated "(!)" are particularly interesting or instructive.

1. (–) We have many tables and many chairs. Let t be the number of tables, d let c be the number of chairs. Write down an inequality that means "We have least four times as many chairs as tables."

2 (–) Fill in the blanks. The equation $x^2 + bx + c = 0$ has exactly one solution en ________________, and it has no solutions when ________________.

3 (–) Given that $x + y = 100$, what is the maximum value of xy?

4 (–) Explain why the square has the largest area among all rectangles with n perimeter.

5 (–) Consider the Celsius (C) and Fahrenheit (F) temperature scales.

C	0	5	10	15	20	25	30
F	32	41	50	59	68	77	86

ress the sentence "The temperature was 10° C and increased by 20° C" using Fahrenheit scale.

6) At a given moment, let f and c be the values of the temperature on Fahrenheit and Celsius scales, respectively. These values are related by

$f = (9/5)c + 32$. At what temperatures do the following events occur?

a) The Fahrenheit and Celsius values of the temperature are equal.
b) The Fahrenheit value is the negative of the Celsius value.
c) The Fahrenheit value is twice the Celsius value.

1.7. (–) The statement below is not always true for $x, y \in \mathbb{R}$. Give an example where it is false, and add a hypothesis on y that makes it a true statement.

"If x and y are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."

1.8. (!) In the morning section of a calculus course, 2 of the 9 women and 2 of the 10 men receive the grade of A. In the afternoon section, 6 of the 9 women and 9 of the 14 men receive A. Verify that, in each section, a higher proportion of women than of men receive A, but that, in the combined course, a lower proportion of women than of men receive A. Explain! (See Exercises 9.19–9.20 for related exercises and Example 9.20 for a real-world example.)

1.9. (–) If a stock declines 20% in one year and rises 23% in the next, is there a net profit? What if it goes up 20% in the first year and down 18% in the next?

1.10. (–) On July 4, 1995, the *New York Times* reported that the nation's universities were awarding 25% more Ph.D. degrees than the economy could absorb. The headline concluded that there was a 1 in 4 chance of underemployment. Here "underemployment" means having no job or having a job not requiring the Ph.D. degree. What should the correct statement of the odds have been?

1.11. (–) A store offers a 15% promotional discount for its grand opening. The clerk believes that the law requires the discount to be applied first and then the tax computed on the resulting amount. A customer argues that the discount should be applied to the total after the 5% sales tax is added, expecting to save more money that way. Does it matter? Explain.

1.12. (–) A store offers an "installment plan" option, with no interest to be paid. There are 13 monthly payments, with the first being a "down payment" that is half the size of the others, so payment is completed one year after purchase. If a customer buys a $1000 stereo, what are the payments under this plan?

1.13. (–) Let A be the set of integers expressible as $2k - 1$ for some $k \in \mathbb{Z}$. Let B be the set of integers expressible as $2k + 1$ for some $k \in \mathbb{Z}$. Prove that $A = B$.

1.14. (–) Let a, b, c, d be real numbers with $a < b < c < d$. Express the set $[a, b] \cup [c, d]$ as the difference of two sets.

1.15. (–) For what conditions on sets A and B does $A - B = B - A$ hold?

1.16. (–) Starting with a single pile of 5 pennies, determine what happens when the operation of Application 1.14 is applied repeatedly. Determine what happens when the initial configuration is a single pile of 6 pennies.

1.17. (–) What are the domain and the image of the absolute value function?

1.18. (–) Determine which real numbers exceed their reciprocals by exactly 1.

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1.19. What are the dimensions of a rectangular carpet with perimeter 48 feet and area 108 square feet? Given positive numbers p and a, under what conditions does there exist a rectangular carpet with perimeter p and area a?

1.20. Suppose that r and s are distinct real solutions of the equation $ax^2+bx+c=0$. In terms of a, b, c, obtain formulas for $r+s$ and rs.

1.21. Let a, b, c be real numbers with $a \neq 0$. Find the flaw in the following "proof" that $-b/2a$ is a solution to $ax^2+bx+c=0$.

Let x and y be solutions to the equation. Subtracting $ay^2+by+c=0$ from $ax^2+bx+c=0$ yields $a(x^2-y^2)+b(x-y)=0$, which we rewrite as $a(x+y)(x-y)+b(x-y)=0$. Hence $a(x+y)+b=0$, and thus $x+y=-b/a$. Since x and y can be any solutions, we can apply this computation letting y have the same value as x. With $y=x$, we obtain $2x=-b/a$, or $x=-b/(2a)$.

1.22. We have two identical glasses. Glass 1 contains x ounces of wine; glass 2 contains x ounces of water ($x \geq 1$). We remove 1 ounce of wine from glass 1 and add it to glass 2. The wine and water in glass 2 mix uniformly. We now remove 1 ounce of liquid from glass 2 and add it to glass 1. Prove that the amount of water in glass 1 is now the same as the amount of wine in glass 2.

1.23. A digital 12-hour clock is defective: the reading for hours is always correct, but the reading for minutes always equals the reading for hours. Determine the minimum number of minutes between possible correct readings of the clock.

1.24. Three people register for a hotel room; the desk clerk charges them \$30. The manager returns and says this was an overcharge, instructing the clerk to return \$5. The clerk takes five \$1 bills, but pockets \$2 as a tip and returns only \$1 to each guest. Of the original \$30 payment, each guest actually paid \$9, and \$2 went to the attendant. What happened to the "missing" dollar?

1.25. A census taker interviews a woman in a house. "Who lives here?" he asks. "My husband and I and my three daughters," she replies. "What are the ages of your daughters?" "The product of their ages is 36 and the sum of their ages is the house number." The census taker looks at the house number, thinks, and says, "You haven't given me enough information to determine the ages." "Oh, you're right," she replies, "Let me also say that my eldest daughter is asleep upstairs." "Ah! Thank you very much!" What are the ages of the daughters? (The problem requires "reasonable" mathematical interpretations of its words.)

1.26. (+) Two mail carriers meet on their routes and have a conversation. A: "I know you have three sons. How old are they?" B: "If you take their ages, expressed in years, and multiply those numbers, the result will equal your age." A: "But that's not enough to tell me the answer!" B: "The sum of these three numbers equals the number of windows in that building." A: "Hmm [pause]. But it's still not enough!" B: "My middle son is red-haired." A: "Ah, now it's clear!" How old are the sons? (Hint: The ambiguity at the earlier stages is needed to determine the solution for the full conversation.) (G. P. Klimov)

1.27. Determine the set of real solutions to $|x/(x+1)| \leq 1$.

1.28. (!) *Application of the AGM Inequality.*

a) Use Proposition 1.4 to prove that $x(c-x)$ is maximized when $x=c/2$.

b) For $a>0$, use part (a) to find the value of y maximizing $y(c-ay)$.

1.29. Let x, y, z be nonnegative real numbers such that $y+z \geq 2$. Prove that $(x+y+z)^2 \geq 4x+4yz$. Determine when equality holds.

1.30. (!) Let x, y, u, v be real numbers.

a) Prove that $(xu+yv)^2 \leq (x^2+y^2)(u^2+v^2)$.

b) Determine precisely when equality holds in part (a).

1.31. (+) *Extensions of the AGM Inequality.*

a) Prove that $4xyzw \leq x^4+y^4+z^4+w^4$ for real numbers x, y, z, w. (Hint: Use the inequality $2tu \leq t^2+u^2$ repeatedly.)

b) Prove that $3abc \leq a^3+b^3+c^3$ for nonnegative a, b, c. (Hint: In the inequality of part (a), set w equal to the cube root of xyz.)

1.32. (!) Assuming only arithmetic (not the quadratic formula or calculus), prove that $\{x \in \mathbb{R}\colon x^2-2x-3<0\} = \{x \in \mathbb{R}\colon -1<x<3\}$.

1.33. Let $S=\{(x,y) \in \mathbb{N}^2\colon (2-x)(2+y) > 2(y-x)\}$. Prove that $S=T$, where $T=\{(1,1),(1,2),(1,3),(2,1),(3,1)\}$.

1.34. Let $S=\{(x,y) \in \mathbb{R}^2 : (1-x)(1-y) \geq 1-x-y\}$. Give a simple description of S involving the signs of x and y.

1.35. (!) Determine the set of ordered pairs (x, y) of nonzero real numbers such that $x/y+y/x \geq 2$.

1.36. Let $S=[3]\times[3]$ (the Cartesian product of $\{1,2,3\}$ with itself). Let T be the set of ordered pairs $(x,y) \in \mathbb{Z}\times\mathbb{Z}$ such that $0 \leq 3x+y-4 \leq 8$. Prove that $S \subseteq T$. Does equality hold?

1.37. Determine the set of solutions to the general quadratic inequality $ax^2+bx+c \leq 0$. Express the answer using linear inequalities or intervals. (Use the quadratic formula; the complete solution involves many cases.)

1.38. Let $S=\{x \in \mathbb{R}\colon x(x-1)(x-2)(x-3)<0\}$. Let T be the interval $(0,1)$, and let U be the interval $(2,3)$. Obtain a simple set equality relating S, T, U.

1.39. (!) Given $n \in \mathbb{N}$, let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1<a_2<\cdots<a_n$. Express $\{x \in \mathbb{R}\colon (x-a_1)(x-a_2)\cdots(x-a_n)<0\}$ using the notation for intervals. (For convenience, use $(-\infty, a)$ to denote $\{x \in \mathbb{R}\colon x<a\}$.)

1.40. Let A and B be sets. Explain why the two sets $(A-B)\cup(B-A)$ and $(A\cup B)-(A\cap B)$ must be equal. Check this when A is the set of states in the United States whose names begin with a vowel and B is the set of states in the United States whose names have at most six letters.

1.41. (−) Let A, B, C be sets. Explain the relationships below. Use the definitions of set operations and containment, with Venn diagrams to guide the argument.

a) $A \subseteq A\cup B$, and $A\cap B \subseteq A$.

b) $A-B \subseteq A$.

c) $A\cap B = B\cap A$, and $A\cup B = B\cup A$.

d) $A\subseteq B$ and $B \subseteq C$ imply $A \subseteq C$.

e) $A\cap(B\cap C)=(A\cap B)\cap C$.

f) $A\cup(B\cup C)=(A\cup B)\cup C$.

1.42. Let $A=\{$January, February, …, December$\}$. Given $x \in A$, let $f(x)$ be the number of days in x. Does f define a function from A to $\mathbb{N}$?

1.43. (−) Let $S=\{(x,y)\in\mathbb{R}^2\colon 2x+5y \leq 10\}$. Graph S. Explain how the answer changes when the constraint is $2x+5y<10$.

1.44. (!) Let $S=\{(x,y)\in\mathbb{R}^2\colon x^2+y^2 \leq 100\}$. Let $T=\{(x,y)\in\mathbb{R}^2\colon x+y \leq 14\}$.

a) Graph $S\cap T$.

b) Count the points in $S\cap T$ whose coordinates are both integers.

1.45. (−) Determine whether the rules below define functions from $\mathbb{R}$ to $\mathbb{R}$.
a) $f(x) = |x - 1|$ if $x < 4$ and $f(x) = |x| - 1$ if $x > 2$.
b) $f(x) = |x - 1|$ if $x < 2$ and $f(x) = |x| - 1$ if $x > -1$.
c) $f(x) = ((x+3)^2 - 9)/x$ if $x \neq 0$ and $f(x) = 6$ if $x = 0$.
d) $f(x) = ((x+3)^2 - 9)/x$ if $x > 0$ and $f(x) = x + 6$ if $x < 7$.
e) $f(x) = \sqrt{x^2}$ if $x \geq 2$, $f(x) = x$ if $0 \leq x \leq 4$, and $f(x) = -x$ for $x < 0$.

1.46. Determine the images of the functions $f: \mathbb{R} \to \mathbb{R}$ defined as follows:
a) $f(x) = x^2/(1 + x^2)$. b) $f(x) = x/(1 + |x|)$.

1.47. Let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by $f(a, b) = (a+1)(a+2b)/2$.
a) Show that the image of f is contained in $\mathbb{N}$.
b) (+) Determine exactly which natural numbers are in the image of f. (Hint: Formulate a hypothesis by trying values.)

1.48. Give several descriptions of the function $f: [0, 1] \to [0, 1]$ defined by $f(x) = 1 - x$. Compare with Example 1.29.

1.49. (!) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. For the sum and product of f and g (see Definition 1.25), determine which statements below are true. If true, provide a proof; if false, provide a counterexample.
a) If f and g are bounded, then $f + g$ is bounded.
b) If f and g are bounded, then fg is bounded.
c) If $f + g$ is bounded, then f and g are bounded.
d) If fg is bounded, then f and g are bounded.
e) If both $f + g$ and fg are bounded, then f and g are bounded.

1.50. (!) For S in the domain of a function f, let $f(S) = \{f(x): x \in S\}$. Let C and D be subsets of the domain of f.
a) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.
b) Give an example where equality does not hold in part (a).

.51. When $f: A \to B$ and $S \subseteq B$, we define $I_f(S) = \{x \in A: f(x) \in S\}$. Let X and Y be subsets of B.
a) Determine whether $I_f(X \cup Y)$ must equal $I_f(X) \cup I_f(Y)$.
b) Determine whether $I_f(X \cap Y)$ must equal $I_f(X) \cap I_f(Y)$.
(Hint: Explore this using the schematic representation described in Remark 1.22.)

.52. Let M and N be nonnegative real numbers. Suppose that $|x + y| \leq M$ and $|x - y| \leq N$. Determine the maximum possible value of x as a function of M and N.

.53. Solve Application 1.38 by using inequalities rather than graphs.

.54. (!) Let $S = \{(x, y) \in \mathbb{R}^2: y \leq x \text{ and } x + 3y \geq 8 \text{ and } x \leq 8\}$.
a) Graph the set S.
b) Find the minimum value of $x + y$ such that $(x, y) \in S$. (Hint: On the graph from part (a), sketch the level sets of the function f defined by $f(x, y) = x + y$.)

.55. (+) Let $\mathbf{F}$ be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for $\mathbf{F}$.

.56. (+) Is there a field with exactly four elements? Is there a field with exactly ... elements?